On the chromatic number of random geometric graphs

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Abstract

Given independent random points $X_1, \dots, X_n \in \mathbb{R}^d$ with common probability distribution ν, and a positive distance $r = r(n) > 0$, we construct a random geometric graph G_n with vertex set $\{1, \dots, n\}$ where distinct i and j are adjacent when $\|X_i - X_j\| \le r$. Here $\|.\|$ may be any norm on $\mathbb{R}^d$, and ν may be any probability distribution on $\mathbb{R}^d$ with a bounded density function. We consider the chromatic number $\chi(G_n)$ of G_n and its relation to the clique number $\omega(G_n)$ as $n \to \infty$. Both McDiarmid [12] and Penrose [16] considered the range of r when $r \ll (\frac{\ln n}{n})^{1/d}$ and the range when $r \gg (\frac{\ln n}{n})^{1/d}$, and their results showed a dramatic difference between these two cases. Here we sharpen and extend the earlier results, and in particular we consider the 'phase change' range when $r \sim (\frac{t \ln n}{n})^{1/d}$ with $t > 0$ a fixed constant. Both [12] and [16] asked for the behaviour of the chromatic number in this range. We determine constants $c(t)$ such that $\frac{\chi(G_n)}{nr^d} \to c(t)$ almost surely. Further, we find a "sharp threshold" (except for less interesting choices of the norm when the unit ball tiles d-space): there is a constant $t_0 > 0$ such that if $t \le t_0$ then $\frac{\chi(G_n)}{\omega(G_n)}$ tends to 1 almost surely, but if $t > t_0$ then $\frac{\chi(G_n)}{\omega(G_n)}$ tends to a limit > 1 almost surely.

1 Introduction and statement of results

In this section, after giving some initial definitions including that of the random geometric graphs G_n, we present our main results on colouring these graphs. We then introduce more notation and definitions so that we can specify explicit limits; we discuss fractional chromatic numbers and 'generalised scan statistics', which will be key tools in our proofs; and we sketch the overall plan of the proofs.

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1.1 Some definitions and notation: G_n, δ, σ, χ and ω

To set the stage, we fix a positive integer d and a norm $\|.\|$ on $\mathbb{R}^d$. Given points $x_1, \ldots, x_n$ in $\mathbb{R}^d$ and a 'threshold distance' $r > 0$, the corresponding *geometric graph* $G(x_1, \ldots, x_n; r)$ has vertices $1, \ldots, n$ and distinct vertices i and j are adjacent when $\|x_i - x_j\| \leq r$. It will be convenient sometimes (when the x_i are distinct) to consider $G(x_1, \ldots, x_n; r)$ as having as vertices the points x_i rather than their indices i.

Now introduce a probability distribution ν with bounded density function, and consider a sequence $X_1, X_2, \ldots$ of independent random variables each with this distribution. Also we need a sequence $r = (r(1), r(2), ..)$ of positive real numbers such that $r(n) \to 0$ as $n \to \infty$. The *random geometric graph* G_n is the geometric graph $G(X_1, \ldots, X_n; r(n))$ corresponding to $X_1, \ldots, X_n$ and $r(n)$. Observe that almost surely the X_i are all distinct and we never have $\|X_i - X_j\| = r(n)$; thus it would not really matter if we took the vertex set of G_n as $\{X_1, \ldots, X_n\}$ and said that vertices X_i and X_j (for $i \neq j$) are adjacent when $\|X_i - X_j\| < r(n)$.

The distance $r = r(n)$ plays a role similar to that of the edge-probability $p(n)$ for Erdős-Renyi random graphs $G(n, p)$. Depending on the choice of $r(n)$, qualitatively different types of behaviour can be observed. The various cases are best described in terms of the quantity nr^d, which scales with the average degree of the graph (for a precise result see appendix A of the paper [13] by the second author). We will often refer to the case when $nr^d / \ln n \to 0$ as the *sparse case*, and the case when $nr^d / \ln n \to \infty$ as the *dense case*.

Throughout this paper, we shall use the terminology *almost surely* (or a.s.) in the standard sense from probability theory. That is, a.s. means "with probability one", and if $Z_1, Z_2, \ldots$ are random variables and $c \in \mathbb{R}$ a constant then $Z_n \to c$ a.s. means that $\mathbb{P}(Z_n \to c) = 1$. The notation $f(n) \ll g(n)$ means the same as $f(n) = o(g(n))$, and the notation $f(n) \sim g(n)$ means that $f(n)/g(n) \to 1$ as $n \to \infty$. So in particular, if $Z_1, Z_2, \ldots$ are random variables and $(k_n)_n$ is a sequence of numbers then $Z_n \sim k_n$ a.s. means that $\mathbb{P}(Z_n / k_n \to 1) = 1$.

Before we can state our first result we will need some further notation and definitions. We use σ to denote the essential supremum of the probability density function f of ν, that is

$$\sigma := \sup\{t : \mathrm{vol}(\{x : f(x) > t\}) > 0\}.$$

Here and in the rest of the paper vol(.) denotes the d-dimensional volume (Lebesgue measure). We call σ the *maximum density* of ν.

We also need to define the 'packing density' for the given norm $\|.\|$. Informally this is the greatest proportion of $\mathbb{R}^d$ that can be filled with disjoint translates of the unit ball $B := \{x \in \mathbb{R}^d : \|x\| < 1\}$. For $K > 0$ let $N(K)$ be the maximum cardinality of a collection of pairwise disjoint translates of B with centers in $(0, K)^d$. The (translational) *packing density* δ (of the unit ball B with respect to $\|.\|$) may be defined as

$$\delta := \lim_{K \to \infty} \frac{N(K)\,\mathrm{vol}(B)}{K^d}.$$

This limit always exists, and $0 < \delta \leq 1$. In the case of the Euclidean norm in $\mathbb{R}^2$ we have

$\delta = \frac{\pi}{2\sqrt{3}} \approx 0.907$. For an overview of results on packing see for example the books [14] of Pach and Agarwal or [18] of Rogers.

Recall that a *k-colouring* of a graph G is a map $f : V(G) \to \{1, \ldots, k\}$ such that $f(v) \neq f(w)$ whenever $vw \in E(G)$, and that the *chromatic number* $\chi(G)$ is the least k for which G admits a k-colouring. Also a *clique* in G is a set of vertices which are pairwise adjacent, and the *clique number* $\omega(G)$ is the largest cardinality of a clique (note that $\omega(G)$ is a trivial lower bound for $\chi(G)$). In this paper we are interested mainly in the behaviour of the chromatic number $\chi(G_n)$, and its relation to the clique number $\omega(G_n)$, of the random geometric graph G_n as n grows large.

Assumptions and notation for the random geometric graph G_n. For convenience of reference, we collect our standard assumptions. We assume throughout that we are given a fixed positive integer d and a fixed norm $\|.\|$ on $\mathbb{R}^d$, with packing density δ. Also ν is a probability distribution with finite maximum density σ; $X_1, X_2, \ldots$ are independent random variables each with this distribution; $r = (r(1), r(2), \ldots)$ is a sequence of positive reals such that $r(n) \to 0$ as $n \to \infty$; and for $n = 1, 2, \ldots,$ the *random geometric graph* G_n is the geometric graph $G(X_1, \ldots, X_n; r(n))$.

1.2 Main results

Our first theorem gives quite a full picture of the different behaviours of the chromatic number of the random geometric graph depending on the choice of the sequence r.

Theorem 1.1 *For the random geometric graph G_n as in Section 1.1, the following hold.*

(i) *Suppose that $nr^d \leq n^{-\alpha}$ for some fixed $\alpha > 0$. Then*

$$\mathbb{P}\left(\chi(G_n) \in \left\{ \left\lfloor |\tfrac{\ln n}{\ln(nr^d)}| + \tfrac{1}{2} \right\rfloor, \left\lfloor |\tfrac{\ln n}{\ln(nr^d)}| + \tfrac{1}{2} \right\rfloor + 1 \right\} \text{ for all but finitely many } n \right) = 1.$$

(ii) *Suppose that $n^{-\varepsilon} \ll nr^d \ll \ln n$ for all $\varepsilon > 0$. Then*

$$\chi(G_n) \sim \ln n / \ln\left(\frac{\ln n}{nr^d}\right) \quad \text{a.s.}$$

(iii) *Suppose that $\frac{\sigma n r^d}{\ln n} \to t \in (0, \infty)$. Then*

$$\chi(G_n) \sim f_\chi(t) \cdot \sigma n r^d \quad \text{a.s.},$$

where f_χ is given by (10) below. It depends only on d and $\|.\|$, is continuous and non-increasing; and satisfies $f_\chi(t) \to \frac{\mathrm{vol}(B)}{2^d \delta}$ as $t \to \infty$, and $f_\chi(t) \to \infty$ as $t \downarrow 0$.

(iv) *Suppose that* $nr^d \gg \ln n$ *(but still* $r \to 0$*). Then*

$$\chi(G_n) \sim \frac{\mathrm{vol}(B)}{2^d \delta} \cdot \sigma n r^d \quad a.s.$$

Part **(ii)** can basically already be found in Penrose's book [16], and also in [12] for the case when $d = 2$ and $\|.\|$ is the euclidean norm. We have however settled the minor technical issue of improving the type of convergence in **(ii)** from convergence in probability to almost sure convergence, which was mentioned as an open problem in [16] and [12].

In part **(iv)** we obtain an improvement over a result in Penrose [16], where an almost sure upper bound for $\limsup \frac{\chi(G_n)}{\sigma n r^d}$ of $\frac{\mathrm{vol}(B)}{2^d \delta_L}$ and an almost sure lower bound for $\liminf \frac{\chi(G_n)}{\sigma n r^d}$ of $\frac{\mathrm{vol}(B)}{2^d \delta}$ are given. Here δ_L is the *lattice packing density* of B (that is, the proportion of $\mathbb{R}^d$ that can be filled with disjoint translates of B whose centres are the integer linear combinations of some basis for $\mathbb{R}^d$). The paper [12] by the first author considers only the Euclidean norm in the plane, where δ and δ_L coincide. However, let us note that in general dimension the question of whether $\delta = \delta_L$ is open, even for the Euclidean norm, and it may well be that $\delta > \delta_L$ for some dimensions d.

Part **(iii)** settles the open problem of a "law of large numbers for χ" in this regime posed in [16]. Both Penrose [16] and the first author [12] have studied the chromatic number in the cases when $nr^d \ll \ln n$ and $nr^d \gg \ln n$, but not much was known previously about the behaviour of the chromatic number in the "intermediate" regime when $nr^d = \Theta(\ln n)$. The limiting constant $f_\chi(t)$ is given explicitly by (10) below: since it requires an involved sequence of definitions we defer the precise definition until then.

For comparison to Theorem 1.1 above, we shall now give a result on the clique number in the same flavour as Theorem 1.1. The results listed in Theorem 1.2 below were already shown by Penrose [16]; except that there was an extra assumption that the probability density function of ν has compact support, and in the regime considered in part **(ii)** only convergence in probability was shown and we have added some detail in the regime considered in part **(i)**.

Theorem 1.2 *For the random geometric graph* G_n *as in Section 1.1, the following hold.*

(i) *Suppose that* $nr^d \leq n^{-\alpha}$ *for some fixed* $\alpha > 0$*. Then*

$$\mathbb{P}\left(\omega(G_n) \in \left\{ \left\lfloor \left|\tfrac{\ln n}{\ln(nr^d)}\right| + \tfrac{1}{2} \right\rfloor, \left\lfloor \left|\tfrac{\ln n}{\ln(nr^d)}\right| + \tfrac{1}{2} \right\rfloor + 1 \right\} \text{ for all but finitely many } n \right) = 1.$$

(ii) *Suppose that* $n^{-\varepsilon} \ll nr^d \ll \ln n$ *for all* $\varepsilon > 0$*. Then*

$$\omega(G_n) \sim \ln n / \ln\left(\frac{\ln n}{nr^d}\right) \quad a.s.$$

(iii) *Suppose that* $\frac{\sigma n r^d}{\ln n} \to t \in (0, \infty)$. *Then*

$$\omega(G_n) \sim f_\omega(t) \cdot \sigma n r^d \quad a.s.$$

Here $f_\omega(t)$ *is the unique* $f \geq \frac{\mathrm{vol}(B)}{2^d}$ *that solves* $H(f 2^d / \mathrm{vol}(B)) = \frac{2^d}{t \,\mathrm{vol}(B)}$, *where* $H(x) := x \ln x - x + 1$ *for* $x > 0$. *The function* f_ω *is continuous and strictly decreasing; and satisfies* $f_\omega(t) \to \frac{\mathrm{vol}(B)}{2^d}$ *as* $t \to \infty$, *and* $f_\omega(t) \to \infty$ *as* $t \downarrow 0$.

(iv) *Suppose that* $n r^d \gg \ln n$ *(but still* $r \to 0$*). Then*

$$\omega(G_n) \sim \frac{\mathrm{vol}(B)}{2^d} \cdot \sigma n r^d \quad a.s.$$

Comparing Theorem 1.1 to Theorem 1.2, it is rather striking that when $nr^d \ll \ln n$ the chromatic and clique number have the same behaviour, while when $nr^d \gg \ln n$ they differ by a multiplicative factor (provided $\delta < 1$). Clearly there is some switch in behaviour when $nr^d = \Theta(\ln n)$. The following result shows a threshold phenomenon occurs (provided $\delta < 1$).

Theorem 1.3 *The following hold for* f_χ, f_ω.

(i) *If* $\delta = 1$ *then* $f_\chi(t) = f_\omega(t)$ *for all* $t \in (0, \infty)$.

(ii) *If* $\delta < 1$ *then there exists a constant* $0 < t_0 < \infty$ *such that* $f_\chi(t) = f_\omega(t)$ *for all* $t \leq t_0$ *and the ratio* $f_\chi(t)/f_\omega(t)$ *is continuous and strictly increasing for* $t \geq t_0$.

For $t \in (0, \infty)$ let us write

$$f_{\chi/\omega}(t) := \frac{f_\chi(t)}{f_\omega(t)}. \tag{1}$$

Observe that, by the properties of f_χ and f_ω listed in Theorem 1.1 and Theorem 1.2, the function $f_{\chi/\omega}$ depends only on the choice of d and $\|.\|$, it is continuous on $(0, \infty)$ and

$$\lim_{t \to \infty} f_{\chi/\omega}(t) = \frac{1}{\delta}. \tag{2}$$

By adding a small amount of work to the proofs of Theorem 1.1 and Theorem 1.2 we will also show that

Theorem 1.4 *For the random geometric graph* G_n *as in Section 1.1, with* $r = r(n)$ *an arbitrary sequence that tends to 0, the following holds. Set* $t(n) := \sigma n r^d / \ln n$ *then*

$$\frac{\chi(G_n)}{\omega(G_n)} \sim f_{\chi/\omega}(t(n)) \quad a.s.$$

So in particular, when $\delta < 1$, we see a sharp threshold at $r_0 := (\frac{t_0 \ln n}{\sigma n})^{\frac{1}{d}}$:

Corollary 1.5 *Consider the random geometric graph G_n as in Section 1.1. Suppose $\delta < 1$ and let t_0 be the constant in part* **(ii)** *of Theorem 1.3. If we set $r_0 := (\frac{t_0 \ln n}{\sigma n})^{\frac{1}{d}}$ then the following hold*

(i) *If $\limsup_{n\to\infty} \frac{r}{r_0} \leq 1$ then*

$$\frac{\chi(G_n)}{\omega(G_n)} \to 1 \quad a.s.$$

(ii) *If $\liminf_{n\to\infty} \frac{r}{r_0} > 1$ then*

$$\liminf_{n\to\infty} \frac{\chi(G_n)}{\omega(G_n)} > 1 \quad a.s.$$

In the course of proving Theorem 1.4 we shall prove the following result, which may be of independent interest. It shows that for very small r the clique number and chromatic number are not only concentrated on the same two consecutive integers (as shown by parts **(i)** of Theorems 1.1 and 1.2), but in fact the chromatic number and clique number are equal.

Proposition 1.6 *For the random geometric graph G_n as in Section 1.1, if $nr^d \leq n^{-\alpha}$ for some fixed $\alpha > 0$ then*

$$\mathbb{P}(\chi(G_n) = \omega(G_n) \text{ for all but finitely many } n) = 1.$$

1.3 The weighted integral ξ and explicit limits

For $x > 0$ let $H(x) = \int_1^x \ln y \, dy = x \ln x - x + 1$ as in Theorem 1.2. Observe that $H(1) = 0$ and that the function $H(x)$ is strictly increasing for $x > 1$.

Now let φ be a fixed non-negative, bounded, measurable function with $0 < \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x < \infty$. For $s \geq 0$ let

$$f(s) := \int_{\mathbb{R}^d} H(e^{s\varphi(x)})\mathrm{d}x.$$

It is routine to check that $f(0) = 0$, that $f(s)$ is continuous and strictly increasing in s, that $f(s) < \infty$ for all $s \geq 0$ and that $f(s) \to \infty$ as $s \to \infty$. For $0 < t < \infty$ the *weighting value* $s(\varphi, t)$ is defined to be the unique nonnegative solution s to $f(s) = 1/t$. Observe that the function $s(\varphi, t)$ is strictly decreasing in t, $s(\varphi, t) \to \infty$ as $t \to 0$ and $s(\varphi, t) \to 0$ as $t \to \infty$.

We define $\xi(\varphi, t)$ for $0 < t < \infty$ by

$$\xi(\varphi, t) := \int_{\mathbb{R}^d} \varphi(x) e^{s\varphi(x)} dx \tag{3}$$

where s is the weighting value $s(\varphi, t)$. So far, $\xi(\varphi, t)$ is strictly decreasing in t. It is convenient also to set

$$\xi(\varphi, \infty) := \int \varphi. \tag{4}$$

Further, if $\int \varphi = 0$ (in which case $\varphi = 0$ almost everywhere) then set $\xi(\varphi, t) = 0$ for each $t \in [0, \infty]$, and if $\int \varphi = \infty$ then set $s(\varphi, t) = 0$ and $\xi(\varphi, t) = \infty$ for each $t \in [0, \infty]$. We call $\xi(\varphi, t)$ the *weighted integral* : note that the function ξ depends only on the dimension d.

We may identify $\xi(\varphi, t)$ when $\varphi = 1_W$ for a measurable set $W \subseteq \mathbb{R}^d$ with $0 < \mathrm{vol}(W) < \infty$. For each $w > 0$ define $c(w, t)$ for $t \in (0, \infty]$ as follows: set $c(w, \infty) = w$, and for $0 < t < \infty$ let $c(w, t)$ be the unique solution $x \geq w$ to $H(\frac{x}{w}) = \frac{1}{wt}$. Observe that $c(w, t)$ is continuous and strictly decreasing in t for $t \in (0, \infty)$; and that

$$c(w, t) \to \infty \quad \text{as } t \to 0, \tag{5}$$

and

$$c(w, t) \to w \quad \text{as } t \to \infty. \tag{6}$$

For $0 < t < \infty$ we have $\xi(1_W, t) = e^s \,\mathrm{vol}(W)$ where s is such that $H(e^s)\,\mathrm{vol}(W) = \frac{1}{t}$; and so

$$\xi(1_W, t) = c(\mathrm{vol}(W), t) \tag{7}$$

and this holds also for $t = \infty$ since then both sides equal $\mathrm{vol}(W)$.

Let $B(x; \rho)$ denote the ball $\{y : \|x - y\| < \rho\}$, so that $B = B(0; 1)$. Let us set

$$\varphi_0 := 1_{B(0; \frac{1}{2})}. \tag{8}$$

Observe that the function f_ω in Theorem 1.2 satisfies

$$f_\omega(t) = c(\mathrm{vol}(B(0; 1/2)), t) = \xi(\varphi_0, t). \tag{9}$$

So in particular, by the properties of $c(w, t)$ listed above, f_ω satisfies the properties listed in part **(iii)** of Theorem 1.2.

Call a set $S \subseteq \mathbb{R}^d$ *well-spread* if $\|v - w\| > 1$ for all $v \neq w \in S$; and let $\mathcal{S}$ denote the collection of all such sets. Finally here we call a nonnegative, measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ *(dual) feasible* if it satisfies the condition that $\sum_{v \in S} \varphi(v) \leq 1$ for each set $S \in \mathcal{S}$. For example, φ_0 is feasible. Denote the set of all feasible functions by $\mathcal{F}$. We may now define the real-valued function f_χ on $(0, \infty)$ in Theorem 1.1, by setting

$$f_\chi(t) := \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) \qquad \text{for } \; 0 < t < \infty. \tag{10}$$

Finally note that the real-valued function $f_{\chi/\omega}$ on $(0, \infty)$ defined by (1) satisfies

$$f_{\chi/\omega}(t) = \frac{\sup_{\varphi \in \mathcal{F}} \; \xi(\varphi, t)}{\xi(\varphi_0, t)} \qquad \text{for } \; 0 < t < \infty. \tag{11}$$

1.4 Fractional chromatic number

We shall see that, in Theorem 1.1, the same conclusion holds if we replace $\chi(G)$ by the fractional chromatic number $\chi_f(G)$ and indeed this is the key to the proofs.

Recall that a *stable* or *independent* set in a graph G is a set of vertices which are pairwise non-adjacent, and the chromatic number $\chi(G)$ of G corresponds to a natural integer linear program (ILP), expressing the fact that the chromatic number is the least number of stable sets needed to cover the vertices, as follows. Let A be the vertex-stable set incidence matrix of G, that is, the rows of A are indexed by the vertices v, the columns are indexed by the stable sets S, and $(A)_{v,S} = 1$ if $v \in S$ and $(A)_{v,S} = 0$ otherwise. Then $\chi(G)$ equals

$$\begin{array}{rl} \min & 1^T x \\ \text{subject to} & Ax \geq 1, \\ & x \geq 0, x \text{ integral.} \end{array} \tag{12}$$

The *fractional chromatic number* $\chi_f(G)$ of a graph G is the objective value of the LP-relaxation of (12) (that is, we drop the constraint that x be integral).

It is easy to see that always $\omega(G) \leq \chi_f(G) \leq \chi(G)$. In general both the ratios $\chi_f(G)/\omega(G)$ and $\chi(G)/\chi_f(G)$ can be arbitrarily large (see for example chapter 3 of Scheinerman and Ullman [19]), but that is not the case for geometric graphs (for a given norm on $\mathbb{R}^d$). For $\chi(G) \leq \Delta(G) + 1$ for any graph G, as shown by a natural greedy colouring algorithm; and, since we may cover the unit ball B with a finite number k of sets of diameter < 1, for any geometric graph G we have $\Delta(G) + 1 \leq k\omega(G)$, and so $\omega(G) \leq \chi_f(G) \leq \chi(G) \leq k\omega(G)$. (The *diameter* of a set $A \subseteq \mathbb{R}^d$ is $\sup\{\|x-y\| : x, y \in A\}$.)

For the random geometric graph G_n the two quantities $\chi(G_n)$ and $\chi_f(G_n)$ are even closer. Our approach to proving Theorem 1.1 and the other results will naturally yield:

Theorem 1.7 *For the random geometric graph G_n as in Section 1.1 (with any distance function $r = r(n) = o(1)$), we have $\chi(G_n)/\chi_f(G_n) \to 1$ a.s.*

1.5 Generalised scan statistics

For a set V of points in $\mathbb{R}^d$ and a nonnegative function $\varphi : \mathbb{R}^d \to \mathbb{R}$ we define $M(V, \varphi)$ by:

$$M(V, \varphi) := \sup_{x \in \mathbb{R}^d} \sum_{v \in V} \varphi(v - x).$$

This quantity plays a central role in our analysis, as do the random variables

$$M_\varphi = M_\varphi(n, r) := \sup_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \varphi\left(r^{-1} X_i - x\right)$$

where we have scaled the X_i by r^{-1}. Thus if the points $X_1, \ldots, X_n$ are distinct, and $V = \{r^{-1}X_1, \ldots, r^{-1}X_n\}$ then $M_\varphi(n, r) = M(V, \varphi)$.

Feasible functions φ correspond to feasible solutions to the dual of the LP for the fractional chromatic number. In the special case when φ is the indicator function 1_W of some set $W \subseteq \mathbb{R}^d$, we denote M_φ by M_W. We denote the number of indices $i \in \{1, \ldots, n\}$ such that $X_i \in W$ by $\mathcal{N}(W) = \mathcal{N}_n(W)$; that is, $\mathcal{N}(W) = \sum_{i=1}^n 1_W(X_i)$. (We will often omit the argument or subscript n for the sake of readability.) Notice that M_W is the maximum number of points in any translate of rW; that is $M_W = \max_x \mathcal{N}(x + rW)$. The variable M_W is a *scan statistic* (with respect to the scanning set W), see for example the book [4] by Glaz, Naus and Wallenstein: we call M_φ a *generalised scan statistic*.

We say that a set $W \subseteq \mathbb{R}^d$ has a *small neighbourhood* if it has finite volume and $\lim_{\varepsilon \to 0} \operatorname{vol}(W_\varepsilon) = \operatorname{vol}(W)$, where $W_\varepsilon = W + \varepsilon B$. Then for sets W with finite volume, W has a small neighbourhood if and only if W is bounded and $\operatorname{vol}(\operatorname{cl}(W)) = \operatorname{vol}(W)$, where $\operatorname{cl}(.)$ denotes closure. In particular all compact sets and all bounded convex sets have small neighbourhoods (and the choice of the norm $\|.\|$ is not relevant). We say that a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is *tidy* if it is measurable, bounded, nonnegative, has bounded support and the sets $\{x : \varphi(x) > a\}$ have small neighbourhoods for all $a > 0$.

The proofs of the above theorems rely heavily on the following limiting result concerning the generalised scan statistic M_φ for a tidy function φ.

Theorem 1.8 *Let ν be a probability distribution on $\mathbb{R}^d$ with finite maximum density σ; let $X_1, X_2, \ldots$ be independent random variables each with distribution ν; and let $r = r(n) > 0$ satisfy $r(n) \to 0$ as $n \to \infty$. If φ is a tidy function and if $t(n) := \frac{\sigma n r^d}{\ln n}$ satisfies $\liminf_n t(n) > 0$ then*

$$\frac{M_\varphi}{\sigma n r^d} \sim \xi(\varphi, t(n)) \quad a.s.$$

1.6 Plan of proofs

In the next section, Lemma 2.1 gives basic results on the weighted integral $\xi(\varphi, t)$ which we use throughout the paper. The following section, Section 3, contains the proof of Theorem 1.8 on generalised scan statistics, and includes some more detailed lemmas on this topic which will be needed later. These two sections are quite technical: they could be skipped on a first reading, and referred back to as needed.

In the short Section 4 we give a quick proof of parts **(iii)** and **(iv)** of Theorem 1.2 on $\omega(G_n)$, using Theorem 1.8 (together with Lemma 2.1). In Section 5, we come to the heart of the proofs on $\chi(G_n)$: we first prove some deterministic results on χ and χ_f for geometric graphs, and on feasible functions φ and the weighted integral ξ; and then we deduce parts **(iii)** and **(iv)** of Theorem 1.1 on $\chi(G_n)$ and Theorem 1.3 on $\chi(G_n)/\omega(G_n)$, using Theorem 1.8 (together with Lemma 2.1). In the next section, Section 6, we show that $f_\chi, f_\omega, f_{\chi/\omega}$ and t_0 have the properties claimed in Theorems 1.1, 1.2 and 1.3. Here, as well as using Lemma 2.1 and Theorem 1.8, we bring in some detailed lemmas on generalised scan statistics from Section 3. In Section 7 we complete our proofs, and finally we make some concluding remarks.

2 The weighted integral - basic results

Here we collect some useful observations about the weighted integral $\xi(\varphi, t)$ defined in Section 1.3. Throughout the paper, we will usually omit the domain we are integrating over and simply write $\int \varphi$ instead of $\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x$. All integrals in this paper are over $\mathbb{R}^d$ (and wrt the d-dimensional Lebesgue measure) unless explicitly stated otherwise. The following lemma lists a number of basic properties of $\xi(\varphi, t)$. We will make frequent use of these properties in the rest of the paper.

Lemma 2.1 *Let φ and ψ be non-negative, bounded, integrable functions on $\mathbb{R}^d$, and let $t \in (0, \infty]$.*

(i) *If $\varphi \leq \psi$ then $\xi(\varphi, t) \leq \xi(\psi, t)$.*

(ii) *$\xi(\lambda\varphi, t) = \lambda\, \xi(\varphi, t)$ for any $\lambda > 0$.*

(iii) *$\xi(\varphi + \psi, t) \leq \xi(\varphi, t) + \xi(\psi, t)$.*

(iv) *For $0 < \lambda < 1$ let φ_λ be given by $\varphi_\lambda(x) = \varphi(\lambda x)$. Then $\xi(\varphi, t) \leq \xi(\varphi_\lambda, t) \leq \lambda^{-d}\xi(\varphi, t)$.*

(v) *$\frac{t}{t+h}\, \xi(\varphi, t) \leq \xi(\varphi, t + h) \leq \xi(\varphi, t)$ for $0 < t < \infty$ and $h > 0$.*

(vi) *If $\int \varphi 1_{\{\varphi \geq a\}} \leq \int \psi 1_{\{\psi \geq a\}}$ for all a then $\xi(\varphi, t) \leq \xi(\psi, t)$.*

(vii) *$\xi(\varphi, t) \to \int \varphi = \xi(\varphi, \infty)$ as $t \to \infty$.*

(viii) *Let $\varphi_1, \varphi_2, \ldots$ be non-negative, bounded, integrable functions on $\mathbb{R}^d$, and suppose that $\varphi_n \to \varphi$ pointwise as $n \to \infty$, and $\varphi_n \leq \psi$ for all n. Then $\xi(\varphi_n, t) \to \xi(\varphi, t)$ as $n \to \infty$.*

The case $t = \infty$ is always trivial, so in the proofs we will only consider the case when $t < \infty$. On several occasions, in the proof below and later, we will differentiate an integral over $x \in \mathbb{R}^d$ with respect to a parameter u and swap the order of integration. In all cases this can be justified by means of the fundamental theorem of calculus and Fubini's theorem[1]. A function is *simple* if it takes only finitely many values. We prove the parts of the lemma in a convenient order.

Proof of (vii): If $0 < t \leq t' < \infty$ then $s(\varphi, t) \geq s(\varphi, t')$ so $\xi(\varphi, t) \geq \xi(\varphi, t')$. Thus **(vii)** follows from the monotone convergence theorem. ■

[1] Here we mean the following. If $g(x, u)$ denotes one of $\varphi(x)e^{u\varphi(x)}$ or $H(e^{u\varphi(x)})$ then $\int_{\mathbb{R}^d} g(x, u) - g(x, 0)\mathrm{d}x = \int_{\mathbb{R}^d} \int_0^u g_2(x, w)\mathrm{d}w\mathrm{d}x = \int_0^u \int_{\mathbb{R}^d} g_2(x, w)\mathrm{d}x\mathrm{d}w$, where g_2 denotes the derivative of g wrt. the second argument, and we have used Fubini's theorem to switch the order of integration. Now the fundamental theorem of calculus shows that $\frac{\mathrm{d}}{\mathrm{d}u} \int_{\mathbb{R}^d} g(x, u)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}u} \int_{\mathbb{R}^d} g(x, u) - g(x, 0)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}u} \int_0^u \int_{\mathbb{R}^d} g_2(x, w)\mathrm{d}x\mathrm{d}w = \int_{\mathbb{R}^d} g_2(x, u)\mathrm{d}x$.

Proof of (i): If we differentiate the equation $t\int H(e^{s\varphi}) = 1$ wrt t we find:

$$0 = \int H(e^{s\varphi}) + t\int s's\varphi^2 e^{s\varphi} = \frac{1}{t} + s'st\int \varphi^2 e^{s\varphi},$$

which gives

$$s' = -\frac{1}{t^2 s\int \varphi^2 e^{s\varphi}}.$$

(That s is differentiable can for instance be seen from the implicit function theorem[2].) Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(\varphi,t) = \int s'\varphi^2 e^{s\varphi} = -\frac{1}{t^2 s}.$$

Now notice that $\varphi \le \psi$ implies $s(\varphi,t) \ge s(\psi,t)$, so that for all $0 < t < \infty$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(\varphi,t) \ge \frac{\mathrm{d}}{\mathrm{d}t}\xi(\psi,t),$$

which implies that $\xi(\psi,t) - \xi(\varphi,t)$ is non-increasing. Finally, by **(vii)** we have

$$\lim_{t\to\infty}\left(\xi(\psi,t) - \xi(\varphi,t)\right) = \int\psi - \int\varphi \ge 0,$$

so that $\xi(\psi,t) \ge \xi(\varphi,t)$ for all $t > 0$. ■

Proof of (ii): We must have $s(\lambda\varphi,t) = s(\varphi,t)/\lambda$ as $\int H(e^{s(\varphi,t)\varphi}) = \int H(e^{s(\lambda\varphi,t)\lambda\varphi}) = \frac{1}{t}$. So indeed $\xi(\lambda\varphi,t) = \int \lambda\varphi e^{s(\lambda\varphi,t)\lambda\varphi} = \lambda\int\varphi e^{s(\varphi,t)\varphi} = \lambda\xi(\varphi,t)$. ■

Proof of (viii): First, let $s \ge 0$ be fixed but otherwise arbitrary. Observe that $H(e^{s\varphi_n}) \le H(e^{s\psi})$. Since we also have $\int H(e^{s\psi}) < \infty$ (as observed in section 1.3), the dominated convergence theorem gives

$$\lim_{n\to\infty}\int H(e^{s\varphi_n}) = \int H(e^{s\varphi}).$$

This shows that $\lim_{n\to\infty} s(\varphi_n,t) = s(\varphi,t)$. Thus, for all $\varepsilon > 0$ and n sufficiently large:

$$\int \varphi_n e^{(s(\varphi,t)-\varepsilon)\varphi_n} \le \xi(\varphi_n,t) \le \int\varphi_n e^{(s(\varphi,t)+\varepsilon)\varphi_n} \le \int\psi e^{(s(\varphi,t)+\varepsilon)\psi}.$$

As $\int\psi e^{(s(\varphi,t)+\varepsilon)\psi} < \infty$ the dominated convergence theorem also gives that

$$\int\varphi e^{(s(\varphi,t)-\varepsilon)\varphi} \le \liminf \xi(\varphi_n,t) \le \limsup \xi(\varphi_n,t) \le \int\varphi e^{(s(\varphi,t)+\varepsilon)\varphi}.$$

[2] Set $F(t,s) := \int_{\mathbb{R}^d} H(e^{s\varphi(x)})\mathrm{d}x - \frac{1}{t}$. Observe that $\frac{\partial}{\partial s}F \ne 0$ for all $s > 0$. The implicit function theorem now gives that for every $t > 0$ there are neighbourhoods U of t and V of $s(\varphi,t)$ and a unique function $g : U \to V$ such that $F(t', g(t')) = 0$ for all $t' \in U$; and moreover this g is continuously differentiable. Thus $s(\varphi,t') = g(t')$ for all $t' \in U$ and in particular $s(\varphi,t')$ is differentiable at $t' = t$.

Two more applications of the the dominated convergence theorem now yield

$$\lim_{\varepsilon\to 0}\int \varphi e^{(s(\varphi,t)-\varepsilon)\varphi}=\lim_{\varepsilon\to 0}\int \varphi e^{(s(\varphi,t)+\varepsilon)\varphi}=\xi(\varphi,t),$$

giving the result. ■

Proof of (iii): By **(viii)** it suffices to take φ and ψ as simple functions. What is more, we can assume without loss of generality that $\mathrm{supp}(\varphi)=\mathrm{supp}(\psi)$. Hence we can assume that there are disjoint sets A_i $i=1,\dots,n$ such that $\varphi=\sum_i a_i 1_{A_i}$ and $\psi=\sum_i b_i 1_{A_i}$ where each $a_i>0$ and $b_i>0$. Let $s_0=s(\varphi+\psi,t)$ and define α_i, β_i and γ_i by setting $1/\alpha_i=\int_{A_i}H(e^{s_0 a_i})$, $1/\beta_i=\int_{A_i}H(e^{s_0 b_i})$ and $1/\gamma_i=\int_{A_i}H(e^{s_0(a_i+b_i)})$. Clearly $0<\gamma_i<\alpha_i,\beta_i<\infty$. Denote $(a_i+b_i)1_{A_i}$ by g_i, and note that $s(g_i,\gamma_i)=s_0$. Hence

$$\xi(\varphi+\psi,t)=\int(\varphi+\psi)e^{s_0(\varphi+\psi)}=\sum_i\int g_i e^{s_0 g_i}=\sum_i\xi(g_i,\gamma_i).$$

But now by **(ii)**

$$\begin{aligned}\xi(\varphi+\psi,t) &= \sum_i a_i\xi(1_{A_i},\gamma_i)+\sum_i b_i\xi(1_{A_i},\gamma_i)\\ &\le \sum_i a_i\xi(1_{A_i},\alpha_i)+\sum_i b_i\xi(1_{A_i},\beta_i)\\ &= \xi(\varphi,t)+\xi(\psi,t),\end{aligned}$$

completing the proof. ■

Proof of (vi): It suffices to show that $s(\varphi,t)\ge s(\psi,t)$ for all t, because then the argument given in the proof of part **(i)** will give the result. Therefore it also suffices to show that

$$\int_{\mathbb{R}^d}H(e^{s\varphi(x)})\mathrm{d}x\le\int_{\mathbb{R}^d}H(e^{s\psi(x)})\mathrm{d}x \tag{13}$$

for all $s>0$. It is straightforward to check that $F(y):=\frac{\mathrm{d}}{\mathrm{d}\,y}\left(H(e^{sy})/y\right)\ge 0$ for all $y\ge 0$. But

$$H(e^{sy})=y\int_0^y F(z)\mathrm{d}z=\int_0^\infty F(z)y1_{y\ge z}\mathrm{d}z,$$

and so

$$\int_0^\infty H(e^{s\varphi(x)})\mathrm{d}x=\int_0^\infty\int_0^\infty F(z)\varphi(x)1_{\varphi(x)\ge z}\mathrm{d}z\mathrm{d}x.$$

We may swap the order of integration since all the quantities involved are non-negative. Hence, using the fact that $F(z) \geq 0$

$$\begin{aligned}
\int_0^\infty H(e^{s\varphi(x)})\mathrm{d}x &= \int_0^\infty F(z)\left(\int_0^\infty \varphi(x)1_{\varphi(x)\geq z}\mathrm{d}x\right)\mathrm{d}z \\
&\leq \int_0^\infty F(z)\left(\int_0^\infty \psi(x)1_{\psi(x)\geq z}\mathrm{d}x\right)\mathrm{d}z \\
&= \int_0^\infty H(e^{s\psi(x)})\mathrm{d}x,
\end{aligned}$$

so that (13) holds, as desired. ■

Proof of (iv): Note that the substitution $y = \lambda x$ gives that:

$$\frac{1}{t} = \int_{\mathbb{R}^d} H(e^{s\varphi(\lambda x)})dx = \lambda^{-d}\int_{\mathbb{R}^d} H(e^{s\varphi(y)})dy,$$

so that $s(\varphi_\lambda, t) = s(\varphi, \lambda^{-d}t)$. Using the same substitution we get

$$\xi(\varphi_\lambda, t) = \lambda^{-d}\int_{\mathbb{R}^d} \varphi(y)e^{s(\varphi,\lambda^{-d}t)\varphi(y)}dy = \lambda^{-d}\xi(\varphi, \lambda^{-d}t).$$

The upper bound now follows from the fact that $\lambda^{-d} > 1$ and that $s(\varphi, t)$ is decreasing in t. The lower bound follows from part **(vi)**, because $\int \varphi_\lambda 1_{\{\varphi_\lambda \geq a\}} = \lambda^{-d}\int \varphi 1_{\{\varphi\geq a\}}$ for all a (again by the substitution $y = \lambda x$). ■

Proof of (v): Let $\lambda = \left(\frac{t}{t+h}\right)^{\frac{1}{d}}$. Then $0 < \lambda < 1$ and so by **(iv)** and its proof

$$\xi(\varphi, t) \leq \xi(\varphi_\lambda, t) = \lambda^{-d}\xi(\varphi, \lambda^{-d}t) = \frac{t+h}{t}\xi(\varphi, t+h).$$

Also, we have already seen that $\xi(\varphi, t+h) \leq \xi(\varphi, t)$. ■

We have now completed the proof of Lemma 2.1.

3 Proofs for generalised scan statistics

In this section, after some preliminary results, we consider generalised scan statistics first in the sparse case, then the dense case, and finally prove Theorem 1.8 on the limiting behaviour of M_φ.

Here is a rough sketch of the main idea of the proof of Theorem 1.8, when $\sigma n r^d \sim t \ln n$. Consider first the special case $\varphi = 1_W$. Since there are about r^{-d} disjoint scaled translates rW of W where the probability density is close to σ, we see that M_W behaves like the maximum of about $r^{-d} = n^{1+o(1)}$ independent copies of the number Z of points $X_1, \ldots, X_n$

in a fixed scaled translate rW where the probability density is close to σ; and Z is roughly $\mathrm{Po}(\lambda)$ where $\lambda = \mathrm{vol}(W)\sigma n r^d \sim \mathrm{vol}(W) t \ln n$. Large deviation estimates show that M_W will be about $c\lambda$ where $c > 1$ satisfies $\mathbb{P}(\mathrm{Po}(\lambda) \geq c\lambda) \sim 1/n$; and this happens when $H(c)\lambda \sim \ln n$, that is $H(c)\,\mathrm{vol}(W) t \sim 1$.

For the general case concerning M_φ it suffices to consider a step function $\varphi = \sum_i a_i 1_{A_i}$ where the sets A_i are disjoint. If Z_i corresponds to rA_i just as Z corresponded to rW above, then M_φ behaves like the maximum of about r^{-d} independent copies of $\sum_i a_i Z_i$, and we may proceed as above.

3.1 Preliminaries

We need results on the maximum density σ and disjoint 'dense' sets. The first lemma is from Müller [13], and is a straightforward consequence of the fact that the set of points where ν has density at least $(1-\varepsilon/2)\sigma$ has positive measure.

Lemma 3.1 *Let $W \subseteq \mathbb{R}^d$ be bounded with positive Lebesgue measure and fix $\varepsilon > 0$. Then there exist $\Omega(r^{-d})$ disjoint translates $x_1 + rW, \ldots, x_N + rW$ of rW with $\nu(x_i + rW)/\,\mathrm{vol}(rW) \geq (1-\varepsilon)\sigma$ for all $i = 1, \ldots, N$.*

This last result extends to:

Lemma 3.2 *Fix $\varepsilon > 0$ and let $W \subseteq \mathbb{R}^d$ be bounded and let $W_1, \ldots, W_k$ be a partition of W with $\mathrm{vol}(W_i) > 0$ for all i. Then there exist $\Omega(r^{-d})$ points $x_1, \ldots, x_N$ such that the sets $x_i + rW_j$ are pairwise disjoint and $\nu(x_i + rW_j)/\,\mathrm{vol}(rW_j) > (1-\varepsilon)\sigma$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, k$.*

Proof: Set $p_i := \frac{\mathrm{vol}(W_i)}{\mathrm{vol}(W)}$ and $p := \min_i p_i$. By Lemma 3.1 there exist points $x_1, \ldots, x_N$ with $N = \Omega(r^{-d})$ such that the sets $x_i + rW$ are disjoint and satisfy $\nu(x_i + rW) \geq (1 - p\varepsilon)\sigma\,\mathrm{vol}(rW)$. By construction the sets $x_i + rW_j$ are disjoint. We now observe that $\nu(x_i + rW_j)$ must be $\geq (1-\varepsilon)\sigma\,\mathrm{vol}(rW_j)$, because otherwise

$$
\begin{aligned}
\nu(x_i + rW) &< (1-p_j)\sigma\,\mathrm{vol}(rW) + (1-\varepsilon)p_j\sigma\,\mathrm{vol}(rW) \\
&= (1-p_j\varepsilon)\sigma\,\mathrm{vol}(x_i + rW) \leq \nu(x_i + rW),
\end{aligned}
$$

a contradiction. ∎

For the proofs in this section we will also need some bounds on the binomial, Poisson and multinomial distributions. The following lemma is one of the so-called Chernoff-Hoeffding bounds. A proof can be found for example in Penrose [16].

Lemma 3.3 *Let Z be either binomial or Poisson with $\mu := \mathbb{E}Z > 0$.*

- **(i)** *If $k \geq \mu$ then $\mathbb{P}(Z \geq k) \leq e^{-\mu H(\frac{k}{\mu})}$.*
- **(ii)** *If $k \leq \mu$ then $\mathbb{P}(Z \leq k) \leq e^{-\mu H(\frac{k}{\mu})}$.*

Often the upper bound given by Lemma 3.3 is quite close to the truth. The following lemma gives a lower bound on $\mathbb{P}(\mathrm{Po}(\mu) \geq k)$ which is sufficiently sharp for our purposes (see Penrose [16] for a proof).

Lemma 3.4 *For $k, \mu > 0$ it holds that* $\mathbb{P}(\mathrm{Po}(\mu) = k) \geq \frac{e^{-\frac{1}{12k}}}{\sqrt{2\pi k}} e^{-\mu H(\frac{k}{\mu})}$.

A direct corollary of lemmas 3.3 and 3.4 is the following result:

Lemma 3.5 *For $\alpha > 1$ it holds that* $\mathbb{P}(\mathrm{Po}(\mu) > \alpha\mu) = e^{-\mu H(\alpha) + o(\mu)}$ *as $\mu \to \infty$.*

Another bound on the binomial and Poisson that will be useful in the sequel is the following standard elementary result (see for example McDiarmid [12]).

Lemma 3.6 *Let Z be either binomial or Poisson and $k \geq \mu := \mathbb{E}Z$. Then*

$$(\frac{\mu}{ek})^k \leq \mathbb{P}(Z \geq k) \leq (\frac{e\mu}{k})^k.$$

We will also need the following result from Mallows [10] on the multinomial distribution:

Lemma 3.7 *Let* $(Z_1, \ldots, Z_m) \sim \mathrm{mult}(n; p_1, \ldots, p_m)$. *Then*

$$\mathbb{P}(Z_1 \leq k_1, \ldots, Z_m \leq k_m) \leq \Pi_{i=1}^m \mathbb{P}(Z_i \leq k_i).$$

3.2 Sparse case

Here we consider the behaviour of M_W in the 'very sparse' case, and then the 'quite sparse' case. For suitable sets W we find results on M_W that do not depend on W.

First we introduce a convenient piece of notation. If A is an event then we say that A holds almost surely (a.s.) if $\mathbb{P}(A) = 1$, and if $A_1, A_2, \ldots$ is a sequence of events then $\{A_n$ almost always $\}$ denotes the event that "all but finitely many A_n hold". We will frequently deal with the situation in which $\mathbb{P}(A_n$ almost always $) = 1$, which we shall denote by A_n a.a.a.s. (A_n almost always almost surely). We hope this is a convenient shorthand, which avoids clashes with the many different existing notations for $\mathbb{P}(A_n) = 1 + o(1)$ (a.a., a.a.s., whp.) that are in use in the random graphs literature. The reader should observe that A_n a.a.a.s. is a much stronger statement than A_n a.a.s. Observe that the conclusion of Proposition 1.6 is that in the case considered there we have $\chi(G_n) = \omega(G_n)$ a.a.a.s. We will need the following lemma in order to prove Proposition 1.6.

Lemma 3.8 *Let $W \subseteq \mathbb{R}^d$ be a measurable, bounded set with nonempty interior, and fix $k \in \mathbb{N}$.*

(i) *If $nr^d \leq n^{-\alpha}$ with $\alpha > \frac{1}{k}$ then $M_W \leq k$ a.a.a.s.;*

(ii) *If $nr^d \geq n^{-\beta}$ with $\beta < \frac{1}{k-1}$ then $M_W \geq k$ a.a.a.s.*

Proof of part (i): We first remark that it suffices to prove the result for W a ball. This is because W is bounded and hence we must have $W \subseteq B(0;R)$ for some $R > 0$ (and hence also $M_W \leq M_{B(0;R)}$, so that $M_{B(0;R)} \leq k$ implies $M_W \leq k$). Furthermore, since W is a ball it is clear that M_W is non-decreasing in r and we may assume without loss of generality that r is chosen such that $nr^d = n^{-\alpha}$. If some translate of rW contains $k+1$ points, then some X_i has at least k other points at distance $\leq 2Rr$. Hence

$$\mathbb{P}(M_W \geq k+1) \leq \mathbb{P}(\exists i : \mathcal{N}(B(X_i; 2Rr) \geq k+1) \leq n\mathbb{P}(\mathcal{N}(B(X_1; 2Rr)) \geq k+1).$$

Note that

$$\begin{aligned} \mathbb{P}(\mathcal{N}(B(X_1; 2Rr)) \geq k+1) &\leq \mathbb{P}(\mathrm{Bi}(n, \sigma \operatorname{vol}(B) 2^d R^d r^d) \geq k) \leq (\tfrac{e\sigma \operatorname{vol}(B) 2^d R^d n r^d}{k})^k \\ &= O(n^{-k\alpha}), \end{aligned}$$

where we have used Lemma 3.6. As $\alpha > \frac{1}{k}$, we have $\alpha' := k\alpha - 1 > 0$. We find

$$\mathbb{P}(M_W \geq k+1) = O(n^{-\alpha'}).$$

Unfortunately this expression is not necessarily summable in n so we cannot apply the Borel-Cantelli lemma directly. However, setting $K := \lceil \frac{1}{\alpha'} \rceil + 1$, we may conclude that

$$\mathbb{P}(M_W(m^K, r(m^K)) \leq k \text{ for all but finitely many } m) = 1,$$

because $\sum_m (m^K)^{-\alpha'} < \infty$. We now claim that from this it can be deduced that $M_W \leq k$ a.a.a.s. Note that

$$\lim_{m \to \infty} \frac{r((m-1)^K)}{r(m^K)} = 1,$$

because $nr^d = n^{-\alpha}$. Consequently $\gamma := \sup_m \frac{r((m-1)^K)}{r(m^K)} < \infty$. By the previous we may also conclude that

$$\mathbb{P}(M_{\gamma W}(m^K, r(m^K)) \leq k \text{ for all but finitely many } m) = 1.$$

Let n, m be such that $(m-1)^K < n \leq m^K$. Note that for any $x \in \mathbb{R}^d$ it holds that $x + r(n)W \subseteq x + \gamma r(m^K)W$ as $\gamma r(m^K) \geq r((m-1)^K) > r(n)$. In other words if $M_W(n) \geq k+1$ then also $M_{\gamma W}(m^K) \geq k+1$. Thus it follows that

$$\mathbb{P}(M_W(n, r(n)) \leq k \text{ for all but finitely many } n) = 1,$$

as required. ■

Proof of part (ii): We may assume that $\beta > 0$. Also, we may again assume that W is a ball. This is because W has non-empty interior and it must therefore contain some ball B, so that it suffices to show $M_B \geq k$ a.a.a.s. Again, by the fact that M_W is non-decreasing in r (when W is a ball) we may assume wlog that r is chosen such that $nr^d = n^{-\beta}$. By Lemma 3.1 we can find disjoint translates $W_1, \dots, W_N$ of rW satisfying

$\nu(W_i) \geq (1-\varepsilon)\sigma \operatorname{vol}(W) r^d$ where $N = \Omega(r^{-d})$. Now notice that the joint distribution of $(\mathcal{N}(W_1), \dots, \mathcal{N}(W_N), \mathcal{N}(\mathbb{R}^d \setminus \cup_i W_i))$ is multinomial, so that we can apply Lemma 3.7 to see that

$$\mathbb{P}(M_W \leq k-1) \leq \mathbb{P}(\mathcal{N}(W_1) \leq k-1, \dots, \mathcal{N}(W_N) \leq k-1) \leq \Pi_{i=1}^N \mathbb{P}(\mathcal{N}(W_i) \leq k-1).$$

The (marginal) distribution of $\mathcal{N}(W_i)$ is $\mathrm{Bi}(n, \nu(W_i))$, so that Lemma 3.6 tells us that

$$\mathbb{P}(\mathcal{N}(W_i) \geq k) \geq (\frac{n(1-\varepsilon)\sigma \operatorname{vol}(W) r^d}{ek})^k = cn^{-k\beta}.$$

Thus

$$\mathbb{P}(M_W \leq k-1) \leq (1 - cn^{-k\beta})^N \leq \exp[-cn^{-k\beta} N].$$

As $nr^d = n^{-\beta}$ we have $r^{-d} = n^{1+\beta}$. As $\beta < \frac{1}{k-1}$ we also have $\beta' := 1 + \beta - k\beta > 0$, so that $n^{-k\beta} N = \Omega(n^{\beta'})$. Thus

$$\mathbb{P}(M_W \leq k-1) \leq \exp[-\Omega(n^{\beta'})],$$

which is summable in n. It follows from the Borel-Cantelli lemma that $M_W \geq k$ a.a.a.s. as required. ■

We now move on to consider the 'quite sparse' case.

Lemma 3.9 *Let $W \subseteq \mathbb{R}^d$ be a measurable, bounded set with non-empty interior and fix $0 < \varepsilon < 1$. Then there exists a $\beta = \beta(W, \sigma, \varepsilon) > 0$ such that if $n^{-\beta} \leq nr^d \leq \beta \ln n$ then*

$$(1-\varepsilon)k(n) \leq M_W \leq (1+\varepsilon)k(n) \quad a.a.a.s.,$$

with $k(n) = \ln n / \ln(\frac{\ln n}{nr^d})$.

Proof: As in the proof of the previous lemma we may again assume that W is a ball. Set $k(n) := \ln n / \ln(\frac{\ln n}{nr^d})$. Let us first consider the lower bound. Completely analogously to the proof of Lemma 3.8, part **(ii)**, we have

$$\begin{aligned} \mathbb{P}(M_W < (1-\varepsilon)k) &\leq (1 - \mathbb{P}(\mathrm{Bi}(n, Cr^d) \geq (1-\varepsilon)k))^{\Omega(r^{-d})} \\ &\leq \exp[-\Omega(r^{-d}(\frac{Cnr^d}{e(1-\varepsilon)k})^{(1-\varepsilon)k})] \\ &= \exp[-\Omega(r^{-d}\exp[-(1-\varepsilon)k(\ln(\frac{k}{nr^d}) + D)])], \end{aligned} \tag{14}$$

with $C := (1-\varepsilon)\sigma \operatorname{vol}(W), D := \ln(\frac{e(1-\varepsilon)}{C})$. By choice of k:

$$\begin{aligned} k(\ln(\frac{k}{nr^d}) + D) &= \frac{\ln n}{\ln(\frac{\ln n}{nr^d})} \left[\ln(\frac{\ln n}{nr^d}) - \ln(\ln(\frac{\ln n}{nr^d})) + D\right] \\ &= \ln n \left[1 - \ln(\ln(\frac{\ln n}{nr^d})) / \ln(\frac{\ln n}{nr^d}) + D / \ln(\frac{\ln n}{nr^d})\right]. \end{aligned} \tag{15}$$

If $n^{-\beta} \leq nr^d \leq \beta \ln n$ then $\frac{\ln n}{nr^d} \geq \frac{1}{\beta}$. Thus, if $\beta > 0$ is chosen small enough then:

$$k(\ln(\frac{k}{nr^d}) + D) = \left(1 + \frac{D - \ln(\ln(\frac{\ln n}{nr^d}))}{\ln(\frac{\ln n}{nr^d})}\right) \ln n \leq \ln n. \tag{16}$$

Also note that $r^{-d} \geq \frac{n}{\beta \ln n} = n^{1+o(1)}$. Combining this with (14) and (16), we get

$$\begin{aligned} \mathbb{P}(M_W \leq (1-\varepsilon)k) &\leq \exp[-\Omega(r^{-d}e^{-(1-\varepsilon)\ln n})] = \exp[-\Omega(r^{-d}n^{-1+\varepsilon+o(1)})] \\ &\leq \exp[-n^{\varepsilon+o(1)}]. \end{aligned}$$

This last expression sums in n, so we may conclude that $M_W \geq (1-\varepsilon)k$ a.a.a.s. if $n^{-\beta} \leq nr^d \leq \beta \ln n$ for $\beta > 0$ sufficiently small.

Let us now shift attention to the upper bound. As in the proof of part **(i)** of Lemma 3.8 the obvious upper bound on $\mathbb{P}(M_W \geq (1+\varepsilon)k)$ does not sum in n. Unfortunately the trick we applied there does not seem to work here and we are forced to use a more elaborate method. For $s > 0$ let us set

$$M(n,s) := \max_{x \in \mathbb{R}^d} \mathcal{N}(x + sW), \quad k(n,s) := \ln n / \ln(\frac{\ln n}{ns^d}).$$

Note that $k(n,s)$ is non-decreasing in n and s and so is $M(n,s)$ (because W is a ball). The rough idea for the rest of the proof is as follows. We fix a (large) constant K. Given n we appoximate n by m^K, chosen to satisfy $(m-1)^K < n \leq m^K$, and we approximate r by $\tilde{s} \geq r$, which is one of $O(\ln m)$ candidate values $s_1, \dots, s_{N(m)}$, in such a way that

(a) $M(m^K, \tilde{s}) \leq (1 + \frac{\varepsilon}{2})k(m^K, \tilde{s})$ a.a.a.s.

(b) $(1 + \frac{\varepsilon}{2})k(m^K, \tilde{s}) \leq (1+\varepsilon)k(n,r)$;

Note that $M(m^K, \tilde{s}) \geq M(n, r(n))$ because $m^K \geq n, \tilde{s} \geq r$ and W is a ball, and that combining this with items (a) and (b) will indeed show that $M(n,r) \leq (1+\varepsilon)k$ a.a.a.s. The reason we have chosen this setup is that if the constant K is chosen sufficiently large we will be able to use the Borel-Cantelli lemma to establish (a), making use of the fact that we are only considering a subsequence of $\mathbb{N}$ and $\tilde{s}$ is one of $O(\ln m)$ candidate values.

Let us pick $s_1(n) < s_2(n) < \dots$ such that $k(n, s_i(n)) = i$. Let us denote by $A(n)$ the event

$$A(n) := \{M(n, s_i(n)) > (1 + \frac{\varepsilon}{2})i \text{ for some } 1 \leq i < I(n)\},$$

with $I(n) := \ln n / \ln(\frac{1}{2\beta})$, the value of $k(n,s)$ corresponding to $ns^d = 2\beta \ln n$, where $\beta = \beta(\varepsilon)$ is to be determined later (note that $k(n,s) = i$ implies that $\ln(\frac{\ln n}{ns^d}) = \frac{\ln n}{i}$). By computations done in the proof of **(i)** of Lemma 3.8 we know that

$$\mathbb{P}(M(n,s) > (1+\frac{\varepsilon}{2})k(n,s)) \leq n \left(\frac{Cns^d}{k(n,s)}\right)^{(1+\frac{\varepsilon}{2})k(n,s)} = ne^{-(1+\frac{\varepsilon}{2})k(n,s)(\ln(\frac{k(n,s)}{ns^d})+D)}, \tag{17}$$

for appropriately chosen constants C, D. We may assume wlog that $D \leq 0$. By (15) we have that

$$k(n,s)(\ln\left(\frac{k(n,s)}{ns^d}\right) + D) = \ln n(1 - \frac{\ln\ln(\frac{\ln n}{ns^d})}{\ln(\frac{\ln n}{ns^d})} + \frac{D}{\ln(\frac{\ln n}{ns^d})}).$$

If $s_1 \leq s \leq s_{\lfloor I \rfloor}$ then $\ln n/(ns^d) \geq \frac{1}{2\beta}$. Hence, by taking β sufficiently small we can guarantee that for $s_1 \leq s \leq s_{\lfloor I \rfloor}$:

$$(1+\frac{\varepsilon}{2})(1 - \frac{\ln\ln(\frac{\ln n}{ns^d})}{\ln(\frac{\ln n}{ns^d})} + \frac{D}{\ln(\frac{\ln n}{ns^d})}) \geq (1+\frac{\varepsilon}{2})(1 - \frac{\ln\ln(\frac{1}{2\beta})}{\ln(\frac{1}{2\beta})} + \frac{D}{\ln(\frac{1}{2\beta})}) \geq 1 + \varepsilon/3,$$

since $D \leq 0$. By (17) we have that for $s_1(n) \leq s \leq s_{\lfloor I(n) \rfloor}(n)$:

$$\mathbb{P}(M(n,s) \geq (1+\frac{\varepsilon}{2})k(n,s)) \leq ne^{-(1+\frac{\varepsilon}{2})k(n,s)(\ln(\frac{k(n,s)}{ns^d})+D)} \leq n^{-\varepsilon/3+o(1)}.$$

It also follows that

$$\mathbb{P}(A(n)) \leq I(n)n^{-\varepsilon/3+o(1)} = n^{-\varepsilon/3+o(1)}.$$

This last expression does not necessarily sum in n, but if we take K such that $K\varepsilon/3 > 1$ then we can apply Borel-Cantelli to deduce that (a) holds; that is

$$\mathbb{P}(A(m^K) \text{ holds for at most finitely many } m) = 1.$$

Now let $n \in \mathbb{N}$ be arbitrary and let the integer $m = m(n)$ be such that $(m-1)^K < n \leq m^K$. Let $i = i(n)$ be such that $s_i(m^K) \leq r(n) < s_{i+1}(m^K)$. We first remark that if $n^{-\beta} \leq nr^d \leq \beta \ln n$ then $(m^K)^{-\beta} \leq m^K r^d \leq (\frac{m}{m-1})^K \beta \ln(m^K)$, giving:

$$\frac{1+o(1)}{\beta} \leq k(m^K, r) \leq -(1+o(1))\frac{\ln(m^K)}{\ln(\beta)}.$$

So for n sufficiently large, we must have $\frac{1}{2\beta} < i < I(m^K)$.

To complete the proof we now aim to show that (for n large enough), If $M(n, r(n)) \geq (1+\varepsilon)k(n, r(n))$ then

$$M(m^K, s_{i+1}(m^K)) \geq (1+\frac{\varepsilon}{2})k(m^K, s_{i+1}(m^K)) = (1+\frac{\varepsilon}{2})(i+1).$$

It suffices to show that $(1+\varepsilon)k(n,r(n)) \geq (1+\frac{\varepsilon}{2})(i+1)$, because $M(n,r(n)) \leq M(m^K, s_{i+1}(m^K))$ (since W is a ball). Routine calculations show that

$$k(n, r(n)) \geq k((m-1)^K, s_i(m^K)) \sim k(m^K, s_i(m^K)) = i;$$

and, assuming that $\beta \leq \varepsilon/6$, for n sufficiently large

$$(1+\varepsilon)k(n,r(n)) \geq (1+\frac{\varepsilon}{2})(1+2\beta)i \geq (1+\frac{\varepsilon}{2})(i+1)$$

as required. ■

Lemma 3.9 also allows us to deduce immediately the following corollary, which extends lemma 5.3 of the first author [12] and may be of independent interest.

Lemma 3.10 *Let $W \subseteq \mathbb{R}^d$ be be a measurable, bounded set with non-empty interior. If for each fixed $\varepsilon > 0$ we have $n^{-\varepsilon} < nr^d < \varepsilon \ln n$ for all sufficiently large n, then*

$$M_W \sim \ln n / \ln(\frac{\ln n}{nr^d}) \quad a.s.,$$

3.3 Dense case

The 'dense' case is when $\sigma nr^d / \ln n$ is large. Let us sketch our approach to proving this result. We first show that it suffices to consider a simple function φ. Also we "Poissonise"; that is, we consider random points $X_1, \ldots, X_N$ where N has an appropriate Poisson distribution. To prove the lower bound, we use Lemma 3.2 to see that there are points x_i so that we can translate to regions with density near σ; then we lower bound M_φ by a linear combination of independent Poisson random variables (indeed by independent copies of such linear combinations), and use Lemma 3.3 to complete the proof. The proof of the upper bound is more involved. We first find a suitable simple function $\varphi_\eta \geq \varphi$ and close to φ. Consider a cell $C = [-R/2, R/2)^d$ containing the support of φ_η, and partition $\mathbb{R}^d$ into the sets $\Gamma(x) = x + rR\mathbb{Z}^d$ of translates of the points $x \in rC$. Let the random variable U be uniformly distributed over rC and let $M(U)$ be

$$\max_{y \in \Gamma(U)} \sum_{j=1}^{N} \varphi_\eta(\frac{X_j - y}{r}).$$

Because of the way φ_η was chosen, it suffices to upper bound $\mathbb{P}(M(U) \geq (1+\varepsilon)k)$. To do this we show that for each possible u

$$\mathbb{P}(M(u) \geq (1+\varepsilon)k) = O(r^{-d}) \cdot \mathbb{P}(Z \geq (1+\varepsilon)k)$$

where Z is a linear combination of independent Poisson random variables, and finally we use Lemma 3.3 to complete the proof.

Lemma 3.11 *Let φ be a tidy function. For every $\varepsilon > 0$ there exists a $T = T(\varphi, \varepsilon)$ such that if $\sigma nr^d \geq T \ln n$ then*

$$(1-\varepsilon)k \leq M_\varphi \leq (1+\varepsilon)k \ \ a.a.a.s.,$$

where $k = \sigma nr^d \int \varphi$.

Proof: Let us first observe that it suffices to prove the result for φ a simple function, because the functions φ we are considering can be well approximated by the functions $\varphi_m^{\text{lower}}, \varphi_m^{\text{upper}}$ defined by:

$$\varphi_m^{\text{lower}} := \sum_{k=1}^{\lceil m \cdot \max \varphi \rceil} (\frac{k-1}{m}) 1_{\{\frac{k-1}{m} < \varphi \leq \frac{k}{m}\}}, \quad \varphi_m^{\text{upper}} := \sum_{k=1}^{\lceil m \cdot \max \varphi \rceil} (\frac{k}{m}) 1_{\{\frac{k-1}{m} < \varphi \leq \frac{k}{m}\}},$$

Here we mean by "well approximated" that $\varphi_m^{\text{lower}} \leq \varphi \leq \varphi_m^{\text{upper}}$ for all m and

$$\lim_{m\to\infty} \int \varphi_m^{\text{lower}} = \lim_{m\to\infty} \int \varphi_m^{\text{upper}} = \int \varphi. \tag{18}$$

Observe that (18) follows from the dominated convergence theorem (since φ is bounded and has bounded support). Also observe that the sets $\{\varphi_m^{\text{upper}} > a\} = \{\varphi > \frac{\lfloor am \rfloor}{m}\}$ and $\{\varphi_m^{\text{lower}} > a\} = \{\varphi > \frac{\lceil am \rceil}{m}\}$ have a small neighbourhood for all a.

Clearly $M_{\varphi_m^{\text{lower}}} \leq M_\varphi \leq M_{\varphi_m^{\text{upper}}}$. Thus the result for non-simple functions will follow from the result for simple functions by taking m such that $\int \varphi_m^{\text{upper}} - \int \varphi_m^{\text{lower}} < \frac{\varepsilon}{3} \int \varphi$ and setting $T := \max(T_1, T_2)$, where $T_1 := T(\varphi_m^{\text{upper}}, \frac{\varepsilon}{3})$ is the value we get from the result for simple functions applied to φ_m^{upper} with $\frac{\varepsilon}{3}$ and $T_2 := T(\varphi_m^{\text{lower}}, \frac{\varepsilon}{3})$ is the value we get from the result for simple functions applied to φ_m^{lower} with $\frac{\varepsilon}{3}$.

In the remainder of the proof we will always assume that $\varphi = \sum_{i=1}^m a_i 1_{A_i}$ is a simple function with the sets A_i disjoint and bounded and that $\{\varphi > a\}$ has a small neighbourhood for all a. Let us set

$$k = k(n) := \sigma n r^d \int \varphi. \tag{19}$$

It remains to show that $(1-\varepsilon)k \leq M_\varphi \leq (1+\varepsilon)k$ a.a.a.s., whenever $\sigma n r^d \geq T \ln n$ for some sufficiently large T.

Proof of lower bound: Let $N \sim \text{Po}((1 - \frac{\varepsilon}{100})n)$ be independent from $X_1, X_2, \ldots$. It will be useful to consider $X_1, \ldots, X_N$ (rather than $X_1, \ldots, X_n$), because they constitute the points of a Poisson process with intensity function $(1 - \frac{\varepsilon}{100})nf$ (where f is the probability density function of ν), see for example Kingman [8].

By Lemma 3.2 there are $\Omega(r^{-d})$ points $x_1, \ldots, x_K$ such that $\nu(x_i + rA_j) \geq (1 - \frac{\varepsilon}{100})\sigma \operatorname{vol}(A_j) r^d$ for all $1 \leq i \leq K, 1 \leq j \leq m$ and the sets $x_i + rA_j$ are disjoint. For the current proof we will only need that $K \geq 1$, but the fact that $K = \Omega(r^{-d})$ will be needed for the proof of Theorem 1.8, which proceeds along similar lines to the current proof. Let us set $M_i := \sum_{j=1}^N \varphi(\frac{X_j - x_i}{r})$, so that

$$M_i = a_1 \mathcal{N}_N(x_i + rA_1) + \cdots + a_m \mathcal{N}_N(x_i + rA_m),$$

where $\mathcal{N}_N(B) = \sum_{i=1}^N 1_{X_i \in B}$ denotes the number of points of the Poisson process in B. Note that $\mathcal{N}_N(x_i + rA_j)$ is a Poisson random variable with mean at least

$$\mu_j := (1 - \frac{\varepsilon}{100})^2 \sigma \operatorname{vol}(A_j) n r^d.$$

Setting

$$M'_\varphi := \sup_{x \in \mathbb{R}^d} \sum_{i=1}^N \varphi(\frac{X_i - x}{r}),$$

we have

$$\mathbb{P}(M'_\varphi \leq (1-\varepsilon)k) \leq \mathbb{P}(M_1 \leq (1-\varepsilon)k, \ldots, M_K \leq (1-\varepsilon)k) = \Pi_{i=1}^K \mathbb{P}(M_i \leq (1-\varepsilon)k),$$

where in the last equality we have used that distinct M_i depend on the points of a Poisson process in disjoint areas of $\mathbb{R}^d$ and hence the M_i are independent. If $Z = a_1 Z_1 + \cdots + a_m Z_m$ with the Z_j independent Poisson random variables satisfying $\mathbb{E} Z_j = \mu_j$ then M_i stochastically dominates Z, so that:

$$\mathbb{P}(M_\varphi \le (1-\varepsilon)k) \le \mathbb{P}(M'_\varphi \le (1-\varepsilon)k) + \mathbb{P}(N > n) \le \mathbb{P}(Z \le (1-\varepsilon)k)^K + \mathbb{P}(N > n),$$

and consequently, by Lemma 3.3

$$\mathbb{P}(M_\varphi \le (1-\varepsilon)k) \le \mathbb{P}(Z \le (1-\varepsilon)k)^K + e^{-\alpha n}, \tag{20}$$

where $\alpha := (1 - \frac{\varepsilon}{100}) H(\frac{1}{1-\frac{\varepsilon}{100}})$. But $K = \Omega(r^{-d})$ is ≥ 1 for n sufficiently large, and then $\mathbb{P}(M_\varphi \le (1-\varepsilon)k) \le \mathbb{P}(Z \le (1-\varepsilon)k) + e^{-\alpha n}$. Further, using Lemma 3.3,

$$\begin{array}{rcl} \mathbb{P}(Z \le (1-\varepsilon)k) & \le & \sum_{i=1}^m \mathbb{P}(Z_i \le \frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\mu_i) \\ & \le & m \cdot \max_i \mathbb{P}(\mathrm{Po}(\mu_i) \le \frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\mu_i) \\ & \le & m \cdot \exp[-\min_i \mu_i H\left(\frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\right)]. \end{array}$$

Now suppose that T has been chosen in such a way that (and we may suppose this)

$$T \cdot (1 - \frac{\varepsilon}{100})^2 \cdot \min_i \mathrm{vol}(A_i) \cdot H\left(\frac{1-\varepsilon}{(1-\frac{\varepsilon}{100})^2}\right) \ge 2.$$

It follows that $\sum_n \mathbb{P}(M_\varphi < (1-\varepsilon)k) \le m \sum_n n^{-2} + \sum_n e^{-\alpha n} < \infty$, which concludes the proof of the lower bound.

Proof of upper bound: We may assume wlog that $a_1 > a_2 > \cdots > a_m > 0$. Recall that A_η denotes $A + B(0;\eta) = \cup_{a \in A} B(a;\eta)$. For $\eta > 0$ let φ_η be defined by

$$\varphi_\eta(x) := \left\{ \begin{array}{ll} a_i & \text{if } x \in (A_i)_\eta \setminus \bigcup_{j<i} (A_j)_\eta, \\ 0 & \text{if } x \notin (A_i)_\eta \text{ for all } 1 \le i \le m. \end{array} \right. ,$$

and let η be chosen such that $\int \varphi_\eta \le (1 + \frac{\varepsilon}{100}) \int \varphi$. This can be done, since φ is tidy and so the sets A_i have small neighbourhoods. Thus we can choose η so that

$$\mathrm{vol}((A_i)_\eta \setminus \bigcup_{j<i} (A_j)_\eta) \le (1 + \frac{\varepsilon}{100}) \, \mathrm{vol}(A_i) \tag{21}$$

for all i, and then we also have $\int \varphi_\eta = \sum_i a_i \, \mathrm{vol}((A_i)_\eta \setminus \bigcup_{j<i} (A_j)_\eta) \le (1 + \frac{\varepsilon}{100}) \sum_i a_i \, \mathrm{vol}(A_i) = (1 + \frac{\varepsilon}{100}) \int \varphi$. Clearly $\varphi_\eta(x) \ge \varphi(x)$ for all x giving $M_{\varphi_\eta} \ge M_\varphi$.

Similarly to what we did for the lower bound, let $N \sim \mathrm{Po}((1 + \frac{\varepsilon}{100})n)$ be independent of the X_i and set

$$M'_\varphi := \max_{x \in \mathbb{R}^d} \sum_{j=1}^N \varphi(\frac{X_j - x}{r}).$$

We have

$$\mathbb{P}(M_\varphi > (1+\varepsilon)k) \leq \mathbb{P}(M'_\varphi > (1+\varepsilon)k) + \mathbb{P}(N < n) \leq \mathbb{P}(M'_\varphi > (1+\varepsilon)k) + e^{-\alpha n}, \quad (22)$$

for some $\alpha > 0$ (where we have used the Lemma 3.3). Again the points $X_1, \dots, X_N$ are the points of a Poisson process, this time with intensity function $(1 + \frac{\varepsilon}{100})nf$.

Let $R > 0$ be a fixed constant such that the support of φ_η is contained in $[\frac{-R}{2}, \frac{R}{2})^d$ (R exists because we assumed the A_i are bounded). Let U be uniform on $[0, rR)^d$ and let $\Gamma(U)$ be the random set of points $U + rR\mathbb{Z}^d$ ($= \{U + rRz : z \in \mathbb{Z}^d\}$). For $x \in \mathbb{R}^d$ let M_x be the random variable given by $\sum_{j=1}^N \varphi_\eta(\frac{X_j - x}{r})$. Let us define

$$M(U) := \max_{z \in \Gamma(U)} M_z.$$

If $\|p - q\| \leq \eta r$ then $\varphi_\eta(\frac{x-p}{r}) \geq \varphi(\frac{x-q}{r})$ for all x by definition of φ_η. For any $q \in \mathbb{R}^d$, the probability that some point of $\Gamma(U)$ lies in $B(q; \eta r)$ equals

$$\mathbb{P}(\Gamma(U) \cap B(q; \eta r) \neq \emptyset) = \frac{\mathrm{vol}(B)\eta^d}{R^d}.$$

(We may assume wlog that R is much larger than η.) Because $\sum_{j=1}^N \varphi(\frac{X_j - x}{r}) \leq \sum_{j=1}^N \varphi_\eta(\frac{X_j - y}{r})$ whenever $\|x - y\| < \eta r$, this gives the following inequality:

$$\mathbb{P}(M(U) \geq (1+\varepsilon)k | M'_\varphi \geq (1+\varepsilon)k) \geq \frac{\mathrm{vol}(B)\eta^d}{R^d}.$$

We find:

$$\mathbb{P}(M'_\varphi \geq (1+\varepsilon)k) \leq \frac{R^d}{\mathrm{vol}(B)\eta^d} \mathbb{P}(M(U) \geq (1+\varepsilon)k). \quad (23)$$

Let us now bound $\mathbb{P}(M(U) \geq (1+\varepsilon)k)$. To do this we will condition on $U = u$ and give a uniform bound on $\mathbb{P}(M(u) \geq (1+\varepsilon)k)$. The random variables M_z, $z \in \Gamma(u)$ can be written as $a_1 M_{z,1} + \dots + a_m M_{z,m}$ with the $M_{z,i}$ independent Poisson variables with means

$$\mathbb{E}M_{z,i} \leq (1 + \frac{\varepsilon}{100})^2 \, \mathrm{vol}(A_i)\sigma n r^d =: \mu_i.$$

Let us partition $\Gamma(u)$ into subsets $\Gamma_1, \dots, \Gamma_K$ with $K = O(r^{-d})$ such that

$$\sum_{z \in \Gamma_j} \mathbb{E}M_{z,i} \leq \mu_i \text{ for all } i \in \{1, \dots, m\}. \quad (24)$$

To see that this can be done, notice we can inductively choose maximal subsets $\Gamma_j \subseteq \Gamma(u) \setminus \bigcup_{j' < j} \Gamma_{j'}$ with the property $\sum_{z \in \Gamma_j} \mathbb{E}M_{z,i} \leq \mu_i$ for all $i \in \{1, \dots, m\}$ (where by maximal we mean that the addition to Γ_j of any $z \notin \bigcup_{j' \leq j} \Gamma_{j'}$ would violate this last property). With the Γ_j chosen in this way, we must have that $\Gamma_j \cup \{z\}$ violates one of the

constraints (24) for any $z \in \Gamma_{j+1}$. Thus, in particular $\sum_{i=1}^m \sum_{z\in\Gamma_j\cup\Gamma_{j+1}} \mathbb{E}M_{z,i} > \min_i \mu_i$ if $\Gamma_{j+1} \neq \emptyset$. Consequently, if we were able to select K subsets Γ_j we must have

$$\lfloor\frac{K-1}{2}\rfloor \min_i \mu_i \leq \sum_{j=1}^{K}\sum_{i=1}^{m}\sum_{z\in\Gamma_j} \mathbb{E}M_{z,i} \leq (1+\frac{\varepsilon}{100})n,$$

where the second inequality follows because the $M_{z,i}$ correspond to the number of points of a Poisson process of total intensity $(1+\frac{\varepsilon}{100})n$ in disjoint regions of $\mathbb{R}^d$. So we must indeed have $K = O(r^{-d})$, and that the process of selecting Γ_j must have stopped after $O(r^{-d})$ many Γ_j were selected.

Set $M_{\Gamma_j} := \sum_{z\in\Gamma_j} M_z$. As $\Gamma(u) = \bigcup_j \Gamma_j$ we have

$$M(u) = \max_{z\in\Gamma(u)} M_z \leq \max_j M_{\Gamma_j}.$$

Note the M_{Γ_j} are stochastically dominated by $Z = a_1 Z_1 + \cdots + a_m Z_m$, where the Z_i are independent with $Z_i \sim \mathrm{Po}(\mu_i)$. Thus

$$\mathbb{P}(M(u) \geq (1+\varepsilon)k) \leq K\mathbb{P}(Z \geq (1+\varepsilon)k).$$

Because this bound does not depend on the choice of u we can also conclude

$$\mathbb{P}(M(U) \geq (1+\varepsilon)k) \leq K\mathbb{P}(Z \geq (1+\varepsilon)k). \tag{25}$$

We then have:

$$\begin{aligned}\mathbb{P}(Z \geq (1+\varepsilon)k) &= \mathbb{P}(\textstyle\sum a_i Z_i \geq \frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\sum_i a_i\mu_i) \leq \sum_{i=1}^m \mathbb{P}(Z_i \geq \frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\mu_i)\\ &\leq m\cdot\exp[-\min_i \mu_i H\left(\tfrac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\right)],\end{aligned}$$

using Lemma 3.3. Now suppose that T has been chosen in such a way that (and we may suppose this):

$$T\cdot(1+\frac{\varepsilon}{100})^2\cdot\min_i \mathrm{vol}(A_i)\cdot H\left(\frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\right) \geq 3,$$

so that

$$\exp[-\min_i \mu_i H\left(\frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2}\right)] \leq n^{-3},$$

whenever $\sigma n r^d \geq T\ln n$. Because $K = O(r^{-d})$ and $\sigma n r^d \geq T\ln n$, we have that $K = O(n)$. By (25) we then also have $\mathbb{P}(M(U) \geq (1+\varepsilon)k) = O(n^{-2})$. Combining this with (22) and (23) we find

$$\mathbb{P}(M_\varphi \geq (1+\varepsilon)k) = O(n^{-2}).$$

The Borel-Cantelli lemma now gives the result. ■

3.4 Proof of Theorem 1.8 on M_φ

Our next target will be to prove Theorem 1.8 on the generalised scan statistic M_φ. We will do this along the lines of the proof of Lemma 3.11. We will however need a straightforward generalisation of a Chernoff bound to weighted sums of Poisson variables, which is given by the following lemma.

Lemma 3.12 *Let $X_1, \dots, X_m$ be independent Poisson variables with $X_i \sim \mathrm{Po}(\lambda_i \mu)$ where $\lambda_i > 0$ is fixed, and set $Z := a_1 X_1 + \dots + a_m X_m$ with $a_1, \dots, a_m > 0$ fixed. Then for each fixed $s > 0$, as $\mu \to \infty$*

$$\mathbb{P}(Z \geq \mu \sum_i \lambda_i a_i e^{a_i s}) = \exp\left(-\mu \sum_i \lambda_i H(e^{a_i s}) + o(\mu)\right).$$

Proof: The moment generating function of Z (evaluated at s) is

$$\mathbb{E}e^{sZ} = \Pi_i \mathbb{E}e^{a_i s X_i} = \exp[\sum_i \lambda_i \mu (e^{a_i s} - 1)].$$

Hence Markov's inequality gives

$$\begin{aligned}
\mathbb{P}(Z \geq \mu \sum_i \lambda_i a_i e^{a_i s}) &= \mathbb{P}(e^{sZ} \geq e^{\mu s \sum_i \lambda a_i e^{a_i s}}) \\
&\leq \exp[\mu \sum_i \lambda_i (e^{a_i s} - 1) - \mu s \sum_i \lambda_i a_i e^{a_i s}] \\
&= \exp[-\mu \sum_i \lambda_i (a_i s e^{a_i s} - e^{a_i s} + 1)] \\
&= \exp[-\mu \sum_i \lambda_i H(e^{a_i s}).]
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{P}(Z \geq \mu \sum_i \lambda_i a_i e^{a_i s}) &\geq \mathbb{P}(X_1 \geq \mu \lambda_1 e^{a_1 s}, \dots, X_m \geq \mu \lambda_m e^{a_m s}) \\
&= \exp[-\mu \sum_i \lambda_i H(e^{a_i s}) + o(\mu)],
\end{aligned}$$

using Lemma 3.5. ■

Proof of Theorem 1.8: Suppose first that

$$\frac{\sigma n r^d}{\ln n} \to t \in (0, \infty) \text{ as } n \to \infty. \tag{26}$$

Let us observe that in this case the statement to be proven amounts to

$$\frac{M_\varphi}{\sigma n r^d} \to \xi(\varphi, t) \quad \text{a.s.} \tag{27}$$

We proceed as in the proof of Lemma 3.11. Again it suffices to prove Theorem 1.8 for φ a simple function, because the functions φ considered can be well approximated by the functions $\varphi_m^{\text{lower}}, \varphi_m^{\text{upper}}$ defined in the proof of Lemma 3.11, where this time we mean by "well approximated" that

$$\lim_{m\to\infty} \xi(\varphi_m^{\text{lower}}, t) = \lim_{m\to\infty} \xi(\varphi_m^{\text{upper}}, t) = \xi(\varphi, t). \tag{28}$$

Observe that (28) follows from part **(viii)** of Lemma 2.1 (φ is bounded and has bounded support). So the result for non-simple functions will follow from the result for simple functions by noticing that $M_{\varphi_m^{\text{lower}}} \leq M_\varphi \leq M_{\varphi_m^{\text{upper}}}$ for all m and taking $m \to \infty$.

Now let $\varphi = \sum_{i=1}^m a_i 1_{A_i}$ be a tidy simple function with the sets A_i disjoint, and with $0 < \int \varphi < \infty$. Then $s = s(\varphi, t) > 0$ solves $\int_{\mathbb{R}^d} H(e^{s\varphi(x)})\mathrm{d}x = 1/t$. Note that

$$\begin{aligned}\textstyle\int_{\mathbb{R}^d} \varphi(x)e^{s\varphi(x)}\mathrm{d}x &= \textstyle\sum_{i=1}^m a_i e^{sa_i}\operatorname{vol}(A_i),\\ \textstyle\int_{\mathbb{R}^d} H(e^{s\varphi(x)})\mathrm{d}x &= \textstyle\sum_{i=1}^m H(e^{sa_i})\operatorname{vol}(A_i).\end{aligned}$$

Let us set

$$k := \xi(\varphi, t)\sigma n r^d. \tag{29}$$

Again it suffices to prove that $(1-\varepsilon)k \leq M_\varphi \leq (1+\varepsilon)k$ a.a.a.s., for any $\varepsilon > 0$.

Proof of lower bound in (27): We proceed as in the proof of the lower bound in Lemma 3.11. We restate (20) from there as:

$$\mathbb{P}(M_\varphi \leq (1-\varepsilon)k) \leq \mathbb{P}(Z \leq (1-\varepsilon)k)^K + e^{-\alpha n}, \tag{30}$$

where $\alpha > 0$ is a fixed constant, $K = \Omega(r^{-d})$, and $Z = a_1 Z_1 + \cdots + a_m Z_m$ with the Z_i independent $\mathrm{Po}(\mu_i)$-random variables, where $\mu_i := (1-\frac{\varepsilon}{100})^2\sigma n r^d \operatorname{vol}(A_i)$. We can write

$$\mathbb{P}(Z \leq (1-\varepsilon)k) = \mathbb{P}(Z \leq \frac{(1-\varepsilon)}{(1-\frac{\varepsilon}{100})^2}\sum_{i=1}^m a_i e^{sa_i}\mu_i) = \mathbb{P}(Z \leq \sum_{i=1}^m a_i e^{s'a_i}\mu_i),$$

where $s' = s'(t,\varepsilon)$ solves $\sum_{i=1}^m a_i e^{s'a_i}\operatorname{vol}(A_i) = \frac{(1-\varepsilon)}{(1-\frac{\varepsilon}{100})^2}\sum_{i=1}^m a_i e^{sa_i}\operatorname{vol}(A_i)$. Note that $s' < s$, and (provided ε is small enough) also $s' > 0$. Lemma 3.12 now gives:

$$1-\mathbb{P}(Z \leq (1-\varepsilon)k) = \mathbb{P}(Z > (1-\varepsilon)k) = \exp[-(1-\frac{\varepsilon}{100})^2\sigma n r^d(\sum_{i=1}^m H(e^{a_i s'})\operatorname{vol}(A_i) + o(1))]$$

As $0 < s' < s$ we have that $\sum_{i=1}^m H(e^{a_i s'})\operatorname{vol}(A_i) < \sum_{i=1}^m H(e^{a_i s})\operatorname{vol}(A_i) = \frac{1}{t}$. Consequently there is a constant $c = c(t,\varepsilon) > 0$ such that

$$\mathbb{P}(Z > (1-\varepsilon)k) = \exp[-(1-c+o(1))\ln n] = n^{-1+c+o(1)}.$$

It follows that

$$\mathbb{P}(Z \leq (1-\varepsilon)k)^K \leq (1-n^{-1+c+o(1)})^K \leq \exp[-Kn^{-1+c+o(1)}] \leq \exp[-n^{c+o(1)}],$$

using that K is at least $n^{1+o(1)}$ (as $K = \Omega(r^{-d})$ and $r^{-d} \sim \frac{n}{t \ln n}$), we see that the right hand side of (30) sums in n, so that we may conclude that $M_\varphi \geq (1-\varepsilon)k$ a.a.a.s. by Borel-Cantelli.

Proof of upper bound in (27): Let $N, M'_\varphi, \eta, \varphi_\eta, M(U)$ be as in the proof of the upper bound in Lemma 3.11, where now $\eta > 0$ satisfies $\xi(\varphi_\eta, t) < (1+\varepsilon/100)\xi(\varphi, t)$; and recall from (25) that:

$$\mathbb{P}(M(U) \geq (1+\varepsilon)k) \leq K\mathbb{P}(Z \geq (1+\varepsilon)k),$$

where $K = O(r^{-d})$ and $Z = a_1 Z_1 + \cdots + a_m Z_m$ with the Z_i independent $\mathrm{Po}(\mu_i)$ random variables, where $\mu_i := (1+\frac{\varepsilon}{100})^2 \operatorname{vol}(A_i)\sigma n r^d$. We now have

$$\mathbb{P}(Z \geq (1+\varepsilon)k) = \mathbb{P}(Z \geq \frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2} \sum_i a_i e^{s a_i} \mu_i) = \mathbb{P}(Z \geq \sum_i a_i e^{s' a_i} \mu_i),$$

where $s' = s'(\varepsilon, t)$ is such that $\sum_{i=1}^m a_i e^{s' a_i} \operatorname{vol}(A_i) = \frac{1+\varepsilon}{(1+\frac{\varepsilon}{100})^2} \sum_{i=1}^m a_i e^{s a_i} \operatorname{vol}(A_i)$. Note that $s' > s$, giving $\sum_i H(e^{s' a_i}) \operatorname{vol}(A_i) > \sum_i H(e^{s a_i}) \operatorname{vol}(A_i) = \frac{1}{t}$, and consequently

$$\sum_i H(e^{s' a_i})\mu_i = (1+\frac{\varepsilon}{100})^2 \sigma n r^d \sum_i H(e^{s' a_i}) \operatorname{vol}(A_i) = (1+c+o(1)) \ln n,$$

for some $c = c(\varepsilon, t) > 0$. Since $K = O(r^{-d}) \leq n$ for n large enough we find that:

$$\mathbb{P}(M(U) \geq (1+\varepsilon)k) \leq n\mathbb{P}(Z > (1+\varepsilon)k) = n \exp[-(1+c+o(1)) \ln n] = n^{-c+o(1)}. \tag{31}$$

Unfortunately this does not necessarily sum in n, so we will have to use a more elaborate method than the one used in Lemma 3.11. Note that for any $0 < \eta' < \eta$ we have, completely analogously to (23):

$$\mathbb{P}(M'_{\varphi_{\eta'}} \geq (1+\varepsilon)k) \leq \frac{R^d}{\operatorname{vol}(B)(\eta-\eta')^d} \mathbb{P}(M(U) \geq (1+\varepsilon)k), \tag{32}$$

By (32) and (22) we also have that for all $0 \leq \eta' < \eta$:

$$\mathbb{P}(M_{\varphi_{\eta'}} \geq (1+\varepsilon)k) \leq n^{-c+o(1)} + e^{-\alpha n} = n^{-c+o(1)}.$$

Although the right hand side does not necessarily sum in n, it does hold that if $L > 0$ is such that $cL > 1$ then we can apply the Borel-Cantelli lemma to show that

$$\mathbb{P}(M_{\varphi_{\eta'}}(m^L, r(m^L)) < (1+\varepsilon)k(m^L) \text{ for all but finitely many } m) = 1. \tag{33}$$

We now claim that from this we can conclude that in fact $M_\varphi \leq (1+2\varepsilon)k$ a.a.a.s. To this end, let $n \in \mathbb{N}$ be arbitrary and let $m = m(n)$ be such that $(m-1)^L < n \leq m^L$. The claim follows if we can show that (for n sufficiently large)

$$\{M_{\varphi_{\eta'}}(m^L, r(m^L)) \leq (1+\varepsilon)k(m^L)\} \Rightarrow \{M_\varphi(n) \leq (1+2\varepsilon)k(n)\}. \tag{34}$$

To this end we will first establish that (for n sufficiently large and) for any x, y:

$$\varphi(\frac{y-x}{r(n)}) \leq \varphi_{\eta'}(\frac{y-x}{r(m^L)}). \tag{35}$$

Since the support of φ is contained in $[\frac{-R}{2}, \frac{R}{2}]^d$ we are done if $\|\frac{y-x}{r(n)}\| > \operatorname{diam}([0, \frac{R}{2}]^d) =: \gamma$. If on the other hand $\|\frac{y-x}{r(n)}\| \leq \gamma$ then $\|\frac{y-x}{r(n)} - \frac{y-x}{r(m^L)}\| = |1 - \frac{r(n)}{r(m^L)}| \|\frac{y-x}{r(n)}\| \leq |1 - \frac{r(n)}{r(m^L)}|\gamma = o(1)$ (because $nr^d \sim t \ln n$ giving $r(n) = (1+o(1))r(m^L)$), so that for n sufficiently large this is $< \eta'$ and thus (35) holds uniformly for all x, y (for such sufficiently large n), as required. Since we also have $k(n) = (1+o(1))k(m^L)$, equation (34) does indeed hold for n sufficiently large, which concludes the proof of (27) under the assumption (26) where $t > 0$.

Completing the proof: Now let us drop the assumption (26) and prove the remaining parts of the theorem. Let $t(n) := \sigma nr^d / \ln n$ be as in the statement of Theorem 1.8. Let $\tau = \liminf_n \frac{\sigma nr^d}{\ln n} = \liminf_n t(n)$ so $\tau > 0$ by assumption. Let $0 < \varepsilon < \frac{1}{2}$. We want to show

$$1 - \varepsilon < \liminf_n \frac{M_\varphi}{\sigma nr^d \xi(\varphi, t(n))} \leq \limsup_n \frac{M_\varphi}{\sigma nr^d \xi(\varphi, t(n))} < 1 + \varepsilon \quad \text{a.s.} \tag{36}$$

By Lemma 3.11 and **(vii)** in Lemma 2.1, there is a constant $T < \infty$ such that if $\liminf_n \frac{\sigma nr^d}{\ln n} \geq T$ then (36) holds. To cover the range between τ and T we will use the following claim.

Claim Let $0 < t_a < t_b < \infty$ be such that $\frac{t_b}{t_a} < 1 + \varepsilon$. If $r(n)$ is such that $t_a \leq t(n) \leq t_b$ for n sufficiently large then

$$1 - \varepsilon < \liminf_n \frac{M_\varphi}{\sigma nr^d \xi(\varphi, t(n))} \leq \limsup_n \frac{M_\varphi}{\sigma nr^d \xi(\varphi, t(n))} < 1 + \varepsilon \quad \text{a.s.} \tag{37}$$

Proof of Claim: Let us set

$$n_a = n_a(n) := \lfloor (t_a/t(n)) \cdot n \rfloor, \quad n_b = n_b(n) := \lceil (t_b/t(n)) \cdot n \rceil.$$

By obvious monotonicities

$$M_\varphi(n_a(n), r(n)) \leq M_\varphi(n, r(n)) \leq M_\varphi(n_b(n), r(n))$$

and by part **(v)** of Lemma 2.1

$$\xi(\varphi, t_b) \leq \xi(\varphi, t(n)) \leq \xi(\varphi, t_a) \leq (t_b/t_a)\xi(\varphi, t_b).$$

Hence

$$\frac{M_\varphi(n, r(n))}{\sigma n(r(n))^d \xi(\varphi, t(n))} \geq \frac{M_\varphi(n_a(n), r(n))}{\sigma n(r(n))^d \xi(\varphi, t_a)}, \tag{38}$$

and, since $n_a(n) \sim (t_a/t(n)) \cdot n$, we also have

$$\frac{M_\varphi(n_a(n), r(n))}{\sigma n(r(n))^d \xi(\varphi, t_a)} \sim \frac{M_\varphi(n_a(n), r(n))}{\sigma n_a(n)(r(n))^d \xi(\varphi, t_a)} \cdot \frac{t_a}{t(n)}. \tag{39}$$

Observe that $\sigma n_a(n)(r(n))^d \sim \sigma(t_a/t(n))n(r(n))^d = t_a \ln n \sim t_a \ln n_a(n)$. By the already proved special case of the result under the assumption (26) applied to $M(n_a(n), r(n))$, we therefore have

$$\frac{M_\varphi(n_a(n), r(n))}{\sigma n_a(n)(r(n))^d \xi(\varphi, t_a)} \sim 1 \quad \text{a.s.} \tag{40}$$

To be more explicit, we use the special case twice. Since $\varepsilon < \frac{1}{2}$ and so $\frac{t_a}{t(n)} > \frac{1}{2}$, for each positive integer m the set $\{n : n_a(n) = m\}$ has size 1 or 2. We apply the special case once to $M_\varphi(m, r_L(m))$ and once to $M_\varphi(m, r_U(m))$ where $r_L(m) := \min\{r(n) : n_a(n) = m\}$ and $r_U(m) := \max\{r(n) : n_a(n) = m\}$; and finally we note that $M_\varphi(n_a(n), r(n))$ must be one of $M_\varphi(m, r_L(m))$ or $M_\varphi(m, r_U(m))$ for each n where $m = n_a$.

Hence, combining (38), (39), (40), we also get

$$\liminf_n \frac{M_\varphi}{\sigma n r^d \xi(\varphi, t(n))} \geq \liminf_n \frac{t_a}{t(n)} \geq \frac{t_a}{t_b} > 1 - \varepsilon \quad \text{a.s.}$$

Completely analogously, $\limsup_n \frac{M_\varphi}{\sigma n r^d \xi(\varphi, t(n))} \leq \frac{t_b}{t_a}$ a.s. This completes the proof of the claim (37). ■

Now we put the pieces together. Let m and $t_1 < t_2 < \cdots < t_m$ be such that $t_1 < \tau$, $t_m \geq T$ and $\frac{t_{k+1}}{t_k} < 1 + \varepsilon$ for $k = 1, \ldots, m-1$. For convenience let us also set $t_{m+1} := \infty$. For each $k = 1, \ldots, m$ let

$$r_k(n) := \begin{cases} r(n) & \text{if } \left(\frac{t_k \ln n}{\sigma n}\right)^{\frac{1}{d}} \leq r(n) < \left(\frac{t_{k+1} \ln n}{\sigma n}\right)^{\frac{1}{d}} \\ \left(\frac{t_k \ln n}{\sigma n}\right)^{\frac{1}{d}} & \text{otherwise} \end{cases}$$

Observe that for each n, $M_\varphi(n, r(n))$ equals $M_\varphi(n, r_k(n))$ for some k (that may vary with n). Since each $M_\varphi(n, r_k(n))$ separately satisfies (36) and the intersection of finitely many events of probability one has again probability one, $M_\varphi(n, r)$ itself also satisfies (36). The theorem follows. ■

4 Proof of parts (iii) and (iv) of Theorem 1.2 on $\omega(G_n)$

In this section we use Theorem 1.8 on generalised scan statistics (together with Lemma 2.1) to give a quick proof of the following theorem, which is in a general form that is convenient for the proof of Theorem 1.4 later on.

Theorem 4.1 *If* $t(n) := \frac{\sigma n r^d}{\ln n}$ *satisfies* $\liminf_n t(n) > 0$ *then*

$$\frac{\omega(G_n)}{\sigma n r^d} \sim \xi(\varphi_0, t(n)) \quad \textit{a.s.}$$

Since we have already established in Section 2 that f_ω satisfies the properties claimed by part **(iii)**, this implies parts **(iii)** and **(iv)** of Theorem 1.2.

Proof: Assume that $\liminf_n t(n) > 0$. First set $W := B(0; \frac{1}{2})$. Any set of points contained in a translate of rW is a clique of G_n, so that by Theorem 1.8:

$$\liminf_{n\to\infty} \frac{\omega(G_n)}{\sigma n r^d \xi(\varphi_0, t(n))} \geq \lim_{n\to\infty} \frac{M_W}{\sigma n r^d \xi(\varphi_0, t(n))} = 1 \text{ a.s.}$$

Let us now fix $\varepsilon > 0$ and let $A_1, \dots, A_m \subseteq \varepsilon\mathbb{Z}^d$ be all the subsets of $\varepsilon\mathbb{Z}^d$ that satisfy $0 \in A_i$ and $\operatorname{diam}(A_i) \leq 1 + 2\varepsilon\rho$, where $\rho := \operatorname{diam}([0,1]^d)$. Let $W_i := \operatorname{conv}(A_i)$. We now claim that $\omega(G_n) \leq \max_i M_{W_i}$. To see that this holds, suppose $X_{i_1}, \dots, X_{i_k}$ form a clique in G_n. Let us set $y_j := (X_{i_j} - X_{i_1})/r$ and $A := \{p \in \varepsilon\mathbb{Z}^d : \|p - y_i\| \leq \varepsilon\rho \text{ for some } 1 \leq i \leq k\}$. Observe that $0 = y_1 \in A$ and $\operatorname{diam}(A) \leq 1 + 2\varepsilon\rho$, so that $A = A_i$ for some $1 \leq i \leq m$. What is more $\{y_1, \dots, y_k\} \subseteq W := \operatorname{conv}(A)$. But this gives $\{X_{i_1}, \dots, X_{i_k}\} \subseteq X_{i_1} + rW$ by choice of y_i, and the claim follows.

We will need the Bieberbach inequality, which is sometimes also called the isodiametric inequality. (For a proof of this classical result, see for instance Gruber and Wills [6].)

Lemma 4.2 (Bieberbach inequality) *Let $A \subseteq \mathbb{R}^d$ be measurable and bounded. If A' is a ball with* $\operatorname{diam}(A) = \operatorname{diam}(A')$ *then* $\operatorname{vol}(A) \leq \operatorname{vol}(A')$. ■

By the Bieberbach inequality $\operatorname{vol}(W_i) \leq \operatorname{vol}(B(0; \frac{1+2\varepsilon\rho}{2}))$, so that also $\int \varphi_i 1_{\{\varphi_i \geq a\}} \leq \int \psi 1_{\{\psi \geq a\}}$ for all a where $\varphi_i = 1_{W_i}$ and ψ denotes $1_{B(0; \frac{1+2\varepsilon\rho}{2})}$. Also observe that $\psi(x) = \varphi_0(\frac{x}{1+2\varepsilon\rho})$. By parts **(vi)** and **(iv)** of Lemma 2.1 we therefore have $\max_i \xi(\varphi_i, t) \leq \xi(\psi, t) \leq (1+2\varepsilon\rho)^d \xi(\varphi_0, t)$ for any $t \in (0, \infty)$. Hence for each i we have

$$\frac{M_{W_i}}{\xi(\varphi_0, t(n))} \leq \frac{M_{\varphi_i}}{\xi(\varphi_i, t(n))} \cdot (1 + 2\varepsilon\rho)^d,$$

and so by Theorem 1.8

$$\limsup_n \frac{M_{W_i}}{\xi(\varphi_0, t(n))} \leq (1 + 2\varepsilon\rho)^d \quad \text{a.s.}$$

Now by the claim established above we have $\limsup_n \frac{\omega(G_n)}{\xi(\varphi_0, t(n))} \leq (1 + 2\varepsilon\rho)^d$ a.s. It follows that

$$\frac{\omega(G_n)}{\sigma n r^d \xi(\varphi_0, t(n))} \to 1 \text{ a.s.}$$

which completes the proof. ■

5 Proof of parts (iii) and (iv) of Theorem 1.1

As we mentioned earlier, in Theorem 1.1 the same conclusions will hold if we replace $\chi(G)$ by the fractional chromatic number $\chi_f(G)$,and this is the key to the proof. We first give two deterministic results on χ_f and χ for a geometric graph. Given a finite set $V \subset R^d$, with say $|V| = n$, let us list V arbitrarily as $v_1, \ldots, v_n$; and for $r > 0$ set $G(V, r)$ as the geometric graph $G(v_1, \ldots, v_n; r)$. We are not interested here in the vertex labelling. We show that for each such set V we have $\chi_f(G(V, 1)) = \sup_{\varphi \in \mathcal{F}} M(V, \varphi)$; and then we give an upper bound on $\chi(G(V, 1))$ (from rounding up a solution for χ_f) of the form $(1 + \varepsilon) \max_{i=1,\ldots,m} M(V, \varphi_i) + c$, where the functions φ_i are nearly feasible tidy functions. After that, we give three technical lemmas, and use Theorem 1.8 to complete the proof.

5.1 Deterministic results on fractional chromatic number

In this subsection we give two deterministic results on $\chi_f(G)$ for geometric graphs G. Recall that if $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a function and $V \subseteq \mathbb{R}^d$ is a set of points then $M(V, \varphi) := \sup_{x \in \mathbb{R}^d} \sum_{v \in V} \varphi(v - x)$.

Lemma 5.1 *Let $V \subseteq \mathbb{R}^d$ be a finite set of points and consider the graph $G = G(V, 1)$. Then*

$$\chi_f(G) = \sup_{\varphi \in \mathcal{F}} M(V, \varphi).$$

Proof: Recall that $\chi_f(G)$ is the objective value of the LP relaxation of the integer LP (12). By LP-duality $\chi_f(G)$ also equals the objective value of the dual LP:

$$\begin{array}{rl} \max & 1^T y \\ \text{subject to} & A^T y \leq 1, \\ & y \geq 0. \end{array}$$

For convenience let us write $V = \{v_1, \ldots, v_n\}$ where v_i is the vertex corresponding to the i-th row of A (and thus the i-th column of A^T). Notice that a vector $y = (y_1, \ldots, y_n)^T$ is feasible for the dual LP if and only if it attaches nonnegative weights to the vertices of G in such a way that each stable set has total weight at most one. There is a natural correspondence between such vectors y and certain feasible functions (hence our choice of the name ‘feasible function’).

Let φ be any feasible function and $x \in \mathbb{R}^d$ an arbitrary point. We claim that the vector

$$y = (\varphi(v_1 - x), \ldots, \varphi(v_n - x))^T$$

is a feasible point of the dual LP given above. To see this note that each row of A^T is the incidence vector of some stable set S of G; and $\|(z - x) - (z' - x)\| = \|z - z'\| > 1$ for each $z \neq z' \in S$ since S is stable in G. Hence

$$(A^T y)_S = \sum_{z \in S} \varphi(z - x) \leq 1,$$

by feasibility of φ. This holds for all rows of A^T, so that y is indeed feasible for the dual LP as claimed. Also, notice that the objective function value $1^T y$ equals $\sum_{j=1}^n \varphi(v_j - x)$. This shows that

$$\chi_f(G) \geq \sup_{\varphi \in \mathcal{F}} \sup_{x \in \mathbb{R}^d} \sum_{j=1}^n \varphi(v_j - x) = \sup_{\varphi \in \mathcal{F}} M(V, \varphi).$$

Conversely let the vector $y = (y_1, \ldots, y_n)^T$ be feasible for the dual LP. Define $\varphi(z) = \sum_{i=1}^n y_i 1_{z=v_i}$. Then φ is clearly a feasible function, and

$$1^T y = \sum_{j=1}^n \varphi(v_j) \leq \sup_{x \in \mathbb{R}^d} \sum_{j=1}^n \varphi(v_j - x) = M(V, \varphi),$$

so that $\chi_f(G) \leq \sup_{\varphi \in \mathcal{F}} M(V, \varphi)$. ∎

We now turn our attention towards deriving an upper bound on the chromatic number, and give another deterministic lemma. Given $\alpha > 0$ we say that the function φ on $\mathbb{R}^d$ is α-feasible if the function $\varphi_\alpha(x) = \varphi(\alpha x)$ is feasible (that is, if $S \subseteq \mathbb{R}^d$ satisfies $\|s - s'\| > \alpha$ for all $s \neq s' \in S$ then $\sum_{s \in S} \varphi(s) \leq 1$). Thus 1-feasible means feasible; and if $\alpha < \beta$ and φ is α-feasible then φ is β-feasible.

Lemma 5.2 *For each $\varepsilon > 0$ there exists a positive integer m, simple $(1+\varepsilon)$-feasible, tidy functions $\varphi_1, \ldots, \varphi_m$, and a constant c such that:*

$$\chi(G(V, 1)) \leq (1 + \varepsilon) \max_{i=1,\ldots,m} M(V, \varphi_i) + c,$$

for each finite set $V \subseteq \mathbb{R}^d$

Proof: Let $\varepsilon > 0$, and let $K \in \mathbb{N}$ be a (large) integer. Let us again set $\rho := \mathrm{diam}([0,1]^d)$ and set $L := \lceil (1 + \varepsilon\rho)/\varepsilon \rceil$. Observe that $\|y - z\| \geq 1 + \varepsilon\rho$ whenever $|y_i - z_i| \geq L\varepsilon$ for some coordinate $1 \leq i \leq d$.

We shall show that there exist $(1 + 2\varepsilon\rho)$-feasible tidy functions $\varphi_1, \ldots, \varphi_N$ such that the following holds for any $V \subseteq \mathbb{R}^d$:

$$\chi(G(V, 1)) \leq \left(1 + \frac{L}{2K}\right)^d \max_i M(V, \varphi) + (2K)^{2d} \left(1 + \frac{L}{2K}\right)^d. \tag{41}$$

This of course yields the lemma, by adjusting ε and taking K sufficiently large.

We partition $\mathbb{R}^d$ into hypercubes of side ε. Let Γ be the (infinite) graph with vertex set $\varepsilon\mathbb{Z}^d$ and an edge pq when $\|p - q\| < 1 + \varepsilon\rho$. For each $q \in \varepsilon\mathbb{Z}^d$ let C^q denote the hypercube $q + [0, \varepsilon)^d$. Observe that the hypercubes C^q for $q \in \varepsilon\mathbb{Z}^d$ partition $\mathbb{R}^d$. Thus for each $z \in \mathbb{R}^d$ we may define $p(z)$ to be the unique $q \in \varepsilon\mathbb{Z}^d$ such that $z \in C^q$.

Now let $V_0 = [-K\varepsilon, K\varepsilon)^d \cap \varepsilon\mathbb{Z}^d$, and note that $|V_0| = (2K)^d$. For each $p \in \varepsilon\mathbb{Z}^d$ let Γ^p be the subgraph of Γ induced on the vertex set $p + V_0$, that is by the vertices of Γ in

$p+[-K\varepsilon, K\varepsilon)^d$. Observe that the graphs Γ^p are simply translated copies of Γ^0. Let B be the vertex-stable set incidence matrix of Γ^0.

Now let V be an arbitrary finite subset of $\mathbb{R}^d$ Given a subset S of $\mathbb{R}^d$, let us use the notation $\mathcal{N}(S)$ here to denote $|S \cap V|$. Let Γ_V be the graph we get by replacing each node q of Γ by a clique of size $\mathcal{N}(C^q)$ and adding all the edges between the cliques corresponding to $q, q' \in V_0$ if $qq' \in E(\Gamma^0)$. It is easy to see from the definition of the threshold distance in Γ that $G(V,1)$ is isomorphic to a subgraph of Γ_V. For each $p \in \varepsilon\mathbb{Z}^d$ let Γ_V^p be the subgraph of Γ_V corresponding to the vertices of Γ^p.

Consider some $p \in \varepsilon\mathbb{Z}^d$. Then $\chi(\Gamma_V^p)$ is the objective value of the integer LP:

$$\begin{array}{rl} \min & 1^T x \\ \text{subject to} & Bx \geq b^p \\ & x \geq 0,\ x \text{ integral} \end{array} \tag{42}$$

where $b^p = (\mathcal{N}(C^{p+q}))_{q\in V_0}$ and the vector x is indexed by the stable sets in Γ^0. Here we are using the fact that Γ^p is a copy of Γ^0 and that the vertex corresponding to q has been replaced by a clique of size $\mathcal{N}(C^{p+q})$. By again considering the LP-relaxation and switching to the dual we find that $\chi_f(\Gamma_V^p)$ equals the objective value of the LP:

$$\begin{array}{rl} \max & (b^p)^T y \\ \text{subject to} & B^T y \leq 1 \\ & y \geq 0. \end{array} \tag{43}$$

Notice that the vectors $y = (y_q)_{q\in V_0}$ attach nonnegative weights to the points q of V_0 is such a way that if $S \subseteq V_0$ corresponds to a stable set in Γ_0 then the sum of the weights $\sum_{q\in S} y_q$ is at most one. Note the important fact that the feasible region (that is, the set of all y that satisfy $B^T y \leq 1, y \geq 0$) here does not depend on p or V.

The vectors y that satisfy $B^T y \leq 1, y \geq 0$ correspond to 'nearly feasible' functions φ in a natural way, as follows. Observe that $x \in [-K\varepsilon, K\varepsilon)^d$ if and only if $p(x) \in V_0$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be defined by setting

$$\varphi(x) := \begin{cases} y_{p(x)} & \text{if } x \in [-K\varepsilon, K\varepsilon)^d, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\varphi(x) = \sum_{q\in V_0} 1_{C^q}(x) y_q$. Then for each $p \in \varepsilon\mathbb{Z}^d$

$$\begin{array}{rcl} (b^p)^T y & = & \sum_{q\in V_0} \mathcal{N}(C^{p+q}) y_q = \sum_{q\in V_0} \sum_{v\in V} 1_{C^{p+q}}(v) y_q \\ & = & \sum_{v\in V} \sum_{q\in V_0} 1_{C^q}(v-p) y_q = \sum_{v\in V} \varphi(v-p) \\ & \leq & M(V,\varphi). \end{array}$$

We claim next that the functions φ thus defined are $(1+2\varepsilon\rho)$-feasible; that is they satisfy $\sum_{j=1}^k \varphi(z_j) \leq 1$ for any $z_1, \ldots, z_k$ such that $\|z_j - z_l\| > 1 + 2\varepsilon\rho$ for all $j \neq l$. To see this, pick such $z_1, \ldots, z_k$. Since φ is 0 outside of $[-K\varepsilon, K\varepsilon)^d$ we may as well suppose that all the z_j lie inside $[-K\varepsilon, K\varepsilon)^d$. For $i = 1, \ldots, k$ let $p_i \in V_0$ be the unique point of V_0 such

that $z_i \in p_i + [0, \varepsilon)^d$. For all pairs $i \neq j$ we have $\|p_i - p_j\| \geq \|z_i - z_j\| - \varepsilon\rho > 1 + \varepsilon\rho$. Thus $p_1, \ldots, p_k$ are distinct and form a stable set S in Γ^0, and therefore correspond to one of the rows of B^T. The condition $B^T y \leq 1$ now yields

$$\varphi(z_1) + \cdots + \varphi(z_k) = y_{p_1} + \cdots + y_{p_k} = (B^T y)_S \leq 1.$$

This shows that φ is $(1 + 2\varepsilon\rho)$-feasible as claimed, and it can be readily seen from the definition of φ that it is simple and tidy.

Recall that a basic feasible solution of an LP with k constraints has at most k nonzero elements and that, provided the optimum value is bounded, the optimum value of the LP is always attained at a basic feasible solution (see for example Chvátal [2]). Thus, noting that by rounding up all the variables in an optimum basic feasible solution x to the LP-relaxation of (42) we get a feasible solution of the ILP (42) itself, we see that:

$$\chi(\Gamma_V^p) \leq \chi_f(\Gamma_V^p) + (2K)^d.$$

Now let $y^1, \ldots, y^m$ be the vertices of the polytope $B^T y \leq 1, y \geq 0$ and let $\varphi_1, \ldots, \varphi_m$ be the corresponding $(1 + 2\varepsilon\rho)$-feasible, tidy functions. As the optimum of the LP (43) corresponding to $\chi_f(\Gamma_V^p)$ is attained at one of these vertices we see that:

$$\chi(\Gamma_V^p) \leq \max_{j=1,\ldots,m} (b^p)^T y^j + (2K)^d \leq \max_{j=1,\ldots,m} M(V, \varphi_j) + (2K)^d. \tag{44}$$

What is more, for each $p \in \varepsilon\mathbb{Z}^d$ we can colour any subgraph of $G(V, 1)$ induced by the points in the set

$$W^p := p + [-K\varepsilon, K\varepsilon)^d + (2K + L)\varepsilon\mathbb{Z}^d,$$

with this many colours, since by the definition of $L = \lceil (1+\varepsilon\rho)/\varepsilon \rceil$, the set W^p is the union of hypercubes of side $2K\varepsilon$ which are far enough apart for any two points of Γ in different hypercubes not to be joined by an edge.
Now let the set P be defined by

$$P = \varepsilon\mathbb{Z}^d \cap [-K\varepsilon, (K + L)\varepsilon)^d = \{(\varepsilon i_1, \ldots, \varepsilon i_d) : -K \leq i_j < K + L\}.$$

Note that if p runs through the set P then each $q \in \varepsilon\mathbb{Z}^d$ is covered by exactly $(2K)^d$ of the sets W^p. If H^p is the graph we get by replacing every vertex q of Γ that lies in W^p by a clique of size $\left\lceil \frac{\mathcal{N}(C^q)}{(2K)^d} \right\rceil$ rather than one of size $\mathcal{N}(C^q)$ and removing any vertex that does not lie in W^p, then

$$\chi(H^p) \leq \frac{1}{(2K)^d} \max_j M(V, \varphi_j) + (2K)^d.$$

This is because we can consider the hypercubes of side $2K\varepsilon$ that make up W^p separately (and each of these corresponds to some Γ_V^q) and for each such constituent hypercube $q + [-K\varepsilon, K\varepsilon)^d$ all we need to do is replace the 'right hand side' vector b^q by $\frac{1}{(2K)^d} b^q$ in the LP-relaxation of the ILP (42) (the rounding up of the variables will then take care of

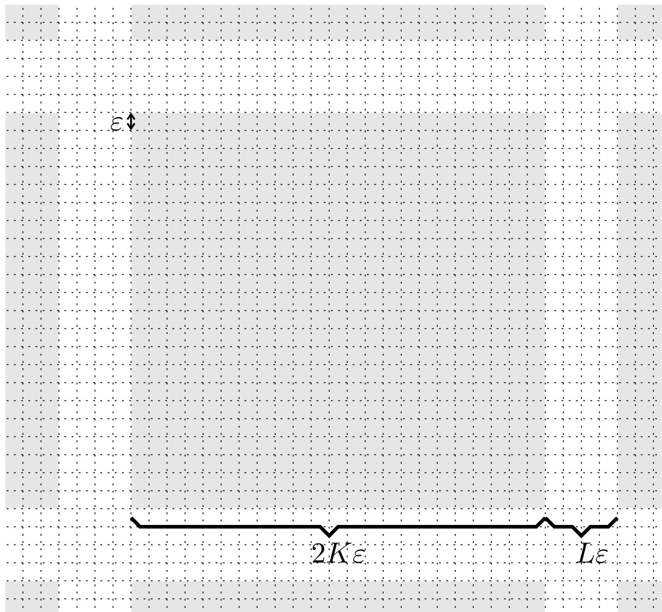

Figure 1: Depiction of a set W^p.

the rounding up in the constraints, as the entries of B are integers). Because each $q \in \varepsilon\mathbb{Z}^d$ is covered by exactly N of the sets W^p we can combine the $(2K+L)^d$ colourings of the graphs H^p for $p \in P$ to get a proper colouring of Γ_V with at most

$$(2K+L)^d \left(\frac{1}{(2K)^d} \max_j M(V, \varphi_j) + (2K)^d \right),$$

colours and the inequality (41) follows. ■

5.2 Some lemmas on $\varphi \in \mathcal{F}$ and $f_\chi(t)$

We now give some lemmas on feasible functions $\varphi \in \mathcal{F}$ and on the function $f_\chi(t) = \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t)$, needed for the proofs in the following subsections.

Lemma 5.3 $\sup_{\varphi\in\mathcal{F}} \int_{\mathbb{R}^d} \varphi(x)dx = \frac{\mathrm{vol}(B)}{2^d\delta}$.

Proof: First note that the function φ_K which has the value $\frac{1}{N(2K)}$ on the hypercube $(0,K)^d$ and 0 elsewhere is feasible, giving that

$$\sup_\varphi \int_{\mathbb{R}^d} \varphi(x)dx \geq \lim_{K\to\infty} \frac{K^d}{N(2K)} = \frac{\mathrm{vol}(B)}{2^d\delta},$$

by definition of the packing constant δ. On the other hand, let φ be an arbitrary feasible function. Let $A \subseteq (0,K)^d$ with $|A| = N(2K)$ be a set of points satisfying $\|a-b\| > 1$ for

all $a \neq b \in A$. If η is a constant such that $\|a - b\| > 1$ whenever $|(a)_i - (b)_i| > \eta$ for some $1 \leq i \leq d$, then the set

$$B := A + (K + \eta)\mathbb{Z}^d \;\; (= \{a + (K + \eta)z : a \in A, z \in \mathbb{Z}^d\})$$

also satisfies the condition that $\|a - b\| > 1$ for all $a \neq b \in B$. Set $\psi(x) := \sum_{b \in B} \varphi(b + x)$. Since φ is feasible we must have $\psi(x) \leq 1$ for all x. For $a \in A$ let us denote by B_a the "coset" $a + (K + \eta)\mathbb{Z}^d \subseteq B$, and let us set $\psi_a(x) := \sum_{b \in B_a} \varphi(b + x)$. We have that

$$(K + \eta)^d \geq \int_{[0, K+\eta)^d} \psi(x)dx = \sum_{a \in A} \int_{[0, K+\eta)^d} \psi_a(x)dx = N(2K) \int_{\mathbb{R}^d} \varphi(x)dx,$$

where the last equality follows because

$$\int_{[0, K+\eta)^d} \psi_a(x)dx = \sum_{b \in B_a} \int_{[0, K+\eta)^d} \varphi(b + x)dx = \sum_{b \in B_a} \int_{b + [0, K+\eta)^d} \varphi(x)dx$$

and the sets $b + [0, K + \eta)^d$ with $b \in B_a$ form a dissection of $\mathbb{R}^d$. Thus we see that indeed for any feasible φ

$$\int_{\mathbb{R}^d} \varphi(x)dx \leq \lim_{K \to \infty} \frac{(K + \eta)^d}{N(2K)} = \frac{\mathrm{vol}(B)}{2^d \delta},$$

as required. ■

From Lemma 5.3 we may conclude:

Lemma 5.4 *Let* $w = \frac{\mathrm{vol}(B)}{2^d \delta}$. *Then* $w \leq \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) \leq c(w, t)$ *for all* $t \in (0, \infty]$.

Proof: The lower bound follows from Lemma 5.3 and the fact that $\xi(\varphi, t) \geq \int \varphi$ (as $s \geq 0$) for all φ. The upper bound follows from Lemma 5.3 together with (7) and part **(vi)** of Lemma 2.1 (if $\varphi \in \mathcal{F}$ and $W \subseteq \mathbb{R}^d$ has $\mathrm{vol}(W) = \frac{\mathrm{vol}(B)}{2^d \delta}$ then $\int \varphi 1_{\{\varphi \geq a\}} \leq \int 1_W 1_{\{1_W \geq a\}}$ for all $a \in \mathbb{R}$ so that $\xi(\varphi, t) \leq \xi(1_W, t)$). ■

Together with observation (5) from section 1.3, Lemma 5.4 implies:

Lemma 5.5 $\lim_{t \to \infty} f_\chi(t) = \frac{\mathrm{vol}(B)}{2^d \delta}$. ■

Moreover, since $\varphi_0 \in \mathcal{F}$ so that $f_\chi(t) = \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) \geq \xi(\varphi_0, t) = f_\omega(t) = c(w, t)$ with $w = \frac{\mathrm{vol}(B)}{2^d}$, observation (6) gives:

Lemma 5.6 $\lim_{t \downarrow 0} f_\chi(t) = \infty$. ■

Since each $\xi(\varphi, t)$ is non-increasing in t for each φ separately, we also have:

Lemma 5.7 $f_\chi(t)$ *is non-increasing.* ■

Observe that, by part **(v)** of Lemma 2.1, for any $h > 0$:

$$(\frac{t}{t+h})\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t) \le \sup_{\varphi\in\mathcal{F}}\xi(\varphi,t+h) \le \sup_{\varphi\in\mathcal{F}}\xi(\varphi,t).$$

Thus:

Lemma 5.8 *$f_\chi(t)$ is continuous in t.* ∎

We shall also need the following two technical lemmas.

Lemma 5.9 *Let $\mathcal{F}^*$ be the collection of all $\varphi \in \mathcal{F}$ that are tidy. For each $0 < t \le \infty$*

$$\sup_{\varphi\in\mathcal{F}^*}\xi(\varphi,t) \;=\; \sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)$$

Proof: If $t = \infty$ then by Lemma 5.3 and the proof of the lower bound in it, both sides of the equation above equal $\frac{\mathrm{vol}(B)}{2^d\delta}$. Thus we may suppose that $0 < t < \infty$. Let $\varphi \in \mathcal{F}$. It suffices to show that

$$\xi(\varphi,t) \le \sup_{\psi\in\mathcal{F}^*}\xi(\psi,t). \tag{45}$$

We may assume that φ has bounded support, because (by Lemma 2.1, part **(viii)**) the sequence of functions $(\varphi_n)_n$ given by $\varphi_n = \varphi 1_{[-n,n]^d}$ satisfies $\lim_{n\to\infty}\xi(\varphi_n,t) = \xi(\varphi,t)$. Let $\varepsilon > 0$ and for each $q \in \varepsilon\mathbb{Z}^d$ let $C^q := q + [0,\varepsilon)^d$ as before. Define the function $\hat\varphi$ on $\mathbb{R}^d$ by setting $\hat\varphi(x) = \sup_{y\in C^{p(x)}}\varphi(y)$ (where again $p(x)$ is the unique $q \in \varepsilon\mathbb{Z}^d$ such that $x \in q+[0,\varepsilon)^d$). Clearly $\hat\varphi \ge \varphi$. Although $\hat\varphi$ is not necessarily feasible, the function φ' given by $\varphi'(x) = \hat\varphi((1+\varepsilon\rho)x)$ is. Also, φ' is tidy: clearly it is measurable, bounded, nonnnegative and has bounded support. That the set $\{\varphi' > a\}$ has a small neighbourhood for all $a > 0$, follows from the fact that it is the union of finitely many hypercubes $(1+\varepsilon\rho)^{-1}C^q$. So $\varphi' \in \mathcal{F}^*$. We find that

$$\xi(\varphi,t) \le \xi(\hat\varphi,t) \le (1+\varepsilon\rho)^d\xi(\varphi',t) \le (1+\varepsilon\rho)^d\sup_{\psi\in\mathcal{F}^*}\xi(\psi,t),$$

using Lemma 2.1, parts **(i)** and **(iv)**, for the first and second inequalities respectively. Now we may send $\varepsilon \to 0$ to conclude the proof of inequality (45), and thus of the lemma. ∎

Lemma 5.10 *Let $0 < \tau < \infty$ and let $\varepsilon > 0$. Then there exist m and tidy functions $\psi_1,\dots,\psi_m$ in $\mathcal{F}$ such that*

$$\max_i\xi(\psi_i,t) \ge (1-\varepsilon)\sup_{\varphi\in\mathcal{F}}\xi(\varphi,t) \quad \textit{for all } t\in[\tau,\infty]. \tag{46}$$

Proof: Recall that each $\xi(\varphi,t)$ is non-increasing as a function of t, and thus so is $f_\chi(t) = \sup_{\varphi\in\mathcal{F}}\xi(\varphi,t)$. Also $f_\chi(\infty) > 0$. Let m and $\tau = \tau_0 < \tau_1 < \cdots < \tau_{m-1} < \tau_m = \infty$ be such that $f_\chi(\tau_i)/f_\chi(\tau_{i+1}) < 1+\varepsilon$ for each $i = 0,1,\dots,m-1$. Let $\psi_1,\dots,\psi_m$ be tidy functions in $\mathcal{F}$ such that $\xi(\psi_i,\tau_i) \ge (1-\varepsilon)f_\chi(\tau_i)$ for each $i = 1,\dots,m$. If $\tau_i \le t \le \tau_{i+1}$ then

$$\xi(\psi_{i+1},t) \ge \xi(\psi_{i+1},\tau_{i+1}) \ge (1-\varepsilon)f_\chi(\tau_{i+1}) \ge (1-\varepsilon)^2 f_\chi(\tau_i) \ge (1-\varepsilon)^2 f_\chi(t).$$

and the lemma follows. ∎

5.3 Completing the proofs of parts (iii) and (iv) of Theorem 1.1

Lemmas 5.5, 5.6, 5.7 and 5.8 show that f_χ has the properties claimed in part **(iii)** of Theorem 1.1.

We shall now prove the following theorem, which implies parts **(iii)** and **(iv)** of Theorem 1.1 and is also convenient for the proof of Theorems 1.4 and 1.7 later on.

Theorem 5.11 *If* $t(n) := \frac{\sigma n r^d}{\ln n}$ *satisfies* $\liminf_n t(n) > 0$ *then*

$$\frac{\chi(G_n)}{\sigma n r^d} \sim f_\chi(t(n)) \quad a.s.$$

Proof: It suffices to prove the following. Assume that $\liminf_n t(n) > 0$ where $t(n) := \frac{\sigma n r^d}{\ln n}$, and let $\varepsilon > 0$. We shall show that a.s.

$$1 - \varepsilon < \liminf_n \frac{\chi_f(G_n)}{\sigma n r^d f_\chi(t(n))} \leq \limsup_n \frac{\chi(G_n)}{\sigma n r^d f_\chi(t(n))} < 1 + \varepsilon. \tag{47}$$

For the lower bound in (47), let $\psi_1, \ldots, \psi_m$ be as in Lemma 5.10. Then, using Lemma 5.1, Theorem 1.8 and Lemma 5.10,

$$\begin{aligned}
\liminf_n \frac{\chi_f(G_n)}{\sigma n r^d f_\chi(t(n))} \quad &\geq \quad \liminf_n \max_i \frac{M_{\psi_i}}{\sigma n r^d \xi(\psi_i, t(n))} \frac{\xi(\psi_i, t(n))}{f_\chi(t(n))} \\
&\geq \quad \liminf_n \max_i \frac{\xi(\psi_i, t(n))}{f_\chi(t(n))} \ \geq \ 1 - \varepsilon \ \text{ a.s.}
\end{aligned}$$

For the upper bound, observe that for $\varphi_1, \ldots, \varphi_m$ and c as in Lemma 5.2, by rescaling we obtain

$$\chi(G_n) = \chi(G(r^{-1}X_1, \ldots, r^{-1}X_n; 1)) \leq (1 + \varepsilon) \max_i M_{\varphi_i} + c.$$

Hence

$$\limsup_n \frac{\chi(G_n)}{\sigma n r^d f_\chi(t(n))} \leq (1 + \varepsilon) \limsup_n \max_i \frac{M_{\varphi_i}}{\sigma n r^d \xi(\varphi_i, t(n))} \leq 1 + \varepsilon \ \text{ a.s.}$$

This completes the proof of (47), and hence of the Theorem. ■

6 Proof of Theorem 1.3

Recall that, since $\varphi_0 = 1_{B(0;\frac{1}{2})} \in \mathcal{F}$ is feasible, and using (9) we have

$$f_\chi(t) = \sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) \geq \xi(\varphi_0, t) = c(w, t) = f_\omega(t),$$

where $w = \frac{\text{vol}(B)}{2^d}$. Thus Lemma 5.4 implies part **(i)** of Theorem 1.3.

In the remainder of the Section 6 we shall thus assume that $\delta < 1$. Let us set

$$t_0 := \inf\{t > 0 : f_{\chi/\omega}(t) \neq 1\}. \tag{48}$$

To achieve our target for this section, we still need to show that $0 < t_0 < \infty$ and that $f_{\chi/\omega}(t)$ is strictly increasing for $t \geq t_0$. We do this in the next subsections.

6.1 Proof that $0 < t_0 < \infty$

Note first that $\xi(\varphi_0, t) > 0$ for all $t > 0$, and by part **(v)** in Lemma 2.1 $\xi(\varphi_0, t)$ is continuous in t. Together with Lemma 5.8 this shows that $f_{\chi/\omega}$ is continuous too.

Since we have already established that $\lim_{t\to\infty} f_\chi(t) = \frac{\mathrm{vol}(B)}{2^d\delta}$ and $\lim_{t\to\infty} f_\omega(t) = \frac{\mathrm{vol}(B)}{2^d}$, we have $\lim_{t\to\infty} f_{\chi/\omega}(t) = \frac{1}{\delta}$. This implies that $t_0 < \infty$ and it therefore only remains to show that $t_0 > 0$.

Let us first give a quick overview of the proof of this section. We first show that $\xi(1_{B(0;3)}, t) < 2\xi(\varphi_0, t)$ holds for all sufficiently small $t > 0$. We then prove that if t satisfies $\xi(1_{B(0;3)}, t) < 2\xi(\varphi_0, t)$ then $f_{\chi/\omega}(t) = 1$. To do this we assume that $f_{\chi/\omega}(t) > 1$, so that there is a feasible function ψ with $\xi(\psi, t) > \xi(\varphi_0, t)$. We show that there must then be such a function which is simple and has support contained in $B(0, 2)$; then we use a convexity argument to replace ψ by a function which takes values only in $\{0, \frac{1}{2}, 1\}$; and then that there must be a function φ_β satisfying $\xi(\varphi_\beta, t) > \xi(\varphi_0, t)$ where φ_β takes a particularly easy form, so that finally we can prove analytically that this cannot happen.

Now we proceed to fill in the details.

Lemma 6.1 *There is a $T > 0$ such that*

$$\xi(1_{B(0;3)}, t) < 2\xi(\varphi_0, t), \tag{49}$$

for all $0 < t \leq T$.

Proof: We shall prove the stronger statement that $\xi(1_{B(0;3)}, t)/\xi(\varphi_0, t) \to 1$ as $t \downarrow 0$. (Observe that this implies the lemma.)

Pick $W \in \{B(0;3), B(0;1/2)\}$, and set $w = \mathrm{vol}(W)$. Recall from section 1.3 that $\xi(1_W, t) = c(w, t)$ where $c(w, t) \geq w$ solves $wH(c/w) = 1/t$. Clearly

$$c(w, t) \to \infty \quad \text{as } t \downarrow 0. \tag{50}$$

Writing out the expression for $wH(c/w)$ we find

$$wH(c/w) = c(w, t)\ln c(w, t) - c(w, t)\ln w - c(w, t) + w. \tag{51}$$

Thus, combining (50) and (51) we see that

$$\frac{1}{t} = wH(c(w, t)/w) = (1 + o(1))c(w, t)\ln c(w, t) \qquad \text{as } t \downarrow 0.$$

But this gives

$$\xi(1_W, t) = (1 + o(1)) \left(\frac{1}{t}\right) / \ln\left(\frac{1}{t}\right) \qquad \text{as } t \downarrow 0. \tag{52}$$

Since (52) is both true for $W = B(0;3)$ and $W = B(0;1/2)$, we see that $\xi(1_{B(0;3)}, t)/\xi(\varphi_0, t) \to 1$ as $t \downarrow 0$, which concludes the proof. ■

Lemma 6.2 *If $t > 0$ satisfies (49) and $\frac{\sigma n r^d}{\ln n} \to t$ as $n \to \infty$, then a.a.a.s. there exists a subgraph H_n of G_n induced by the points in some ball of radius $2r$ such that $\chi(G_n) = \chi(H_n)$.*

Proof: Suppose that t satisfies (49) and $\frac{\sigma n r^d}{\ln n} \to t$ as $n \to \infty$. Let $W = B(0;3)$. By Theorem 1.8 and Theorem 4.1 we have

$$\frac{M_W}{\sigma n r^d} \to \xi(1_W, t) \quad \text{and} \quad \frac{\omega(G_n)}{\sigma n r^d} \to \xi(\varphi_0, t) \quad \text{a.s.}$$

and so $M_W < 2\omega(G_n)$ a.a.a.s. It is convenient to identify vertex i of G_n with X_i. Note that if two vertices with $2r \leq \|X_i - X_j\| \leq 3r$ have degrees $\geq \omega(G_n)$ then there must exist a translate of rW (centred at $\frac{1}{2}(X_i + X_j)$) containing at least $2\omega(G_n) + 2$ points, and so the condition $M_W < 2\omega(G_n)$ fails. Hence a.a.a.s. any two vertices X_i and X_j of G_n with degrees $\geq \omega(G_n)$ satisfy either $\|X_i - X_j\| < 2r$ or $\|X_i - X_j\| > 3r$. But if we remove from G_n all vertices which have degree at most $\omega(G_n) - 1$ then the chromatic number does not change, and by the above we will be left a.a.a.s. with a graph in which each component is contained in some ball of radius $2r$. This completes the proof. ■

We will show that $t_0 \geq T$ (with T as in Lemma 6.1) by proving that $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) = \xi(\varphi_0, t)$ for all t that satisfy (49). The following purely deterministic lemma perhaps surprisingly has a convenient probabilistic proof.

Lemma 6.3 *Let $t > 0$ satisfy* (49). *Then*

$$\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) = \sup_{\substack{\varphi \in \mathcal{F}, \\ \operatorname{supp}(\varphi) \subseteq B(0;2)}} \xi(\varphi, t).$$

Proof: Let r satisfy $\sigma n r^d \sim t \ln n$ and consider $\chi(G_n)$. By Lemma 6.2 a.a.a.s. $\chi(G_n)$ equals the maximum over all $x \in \mathbb{R}^d$ of the chromatic number of the graph induced by the vertices in $B(x;2r)$. Let us fix an $\varepsilon > 0$. Let us denote $V := r^{-1}\{X_1, \ldots, X_n\}$ and let $\Gamma, \Gamma^p, \Gamma_V, \Gamma_V^p$ be as in the proof of Lemma 5.2. For $p \in \varepsilon\mathbb{Z}^d$ let Λ^p denote the subgraph of Γ induced by the points of $\varepsilon\mathbb{Z}^d$ inside $B(p, (2 + \varepsilon\rho))$, where again $\rho := \operatorname{diam}([0,1]^d)$, and let Λ_V^p be the corresponding subgraph of Γ_V^p. Since for every $x \in \mathbb{R}^d$ the subgraph of G_n induced by the vertices inside $B(x, 2r)$ is a subgraph of some Λ_V^p, we have:

$$\chi(G_n) \leq \max_p \chi(\Lambda_V^p) \leq \max_{i=1,\ldots,m} M_{\varphi_i} + c \text{ a.a.a.s.}, \tag{53}$$

where $\varphi_1, \ldots, \varphi_m$ are obtained from the ILP formulation of $\chi(\Lambda_V^p)$ via the same procedure we used in the proof of Lemma 5.2 (that is, the upper bound in (53) is the analogue of the

upper bound in (44)) and c is a constant that depends only on ε, d and $\|.\|$. By construction we have that $\mathrm{supp}(\varphi_i) \subseteq B(0, 2+2\varepsilon\rho)$ and that φ_i' given by $\varphi_i'(x) = \varphi_i((1+2\varepsilon\rho)x)$ is feasible. Notice that φ_i' also satisfies $\mathrm{supp}(\varphi_i') \subseteq B(0;2)$. Thus, (53) together with Theorem 5.11, Theorem 1.8 and part **(iv)** of Lemma 2.1 shows that

$$\sup_{\varphi\in\mathcal{F}} \xi(\varphi,t) \leq \max_{i=1,\dots,m} \xi(\varphi_i,t) \leq \max_{i=1,\dots,m} (1+2\varepsilon\rho)^d \xi(\varphi_i',t) \leq (1+2\varepsilon\rho)^d \sup_{\substack{\varphi\in\mathcal{F},\\ \mathrm{supp}(\varphi)\subseteq B(0;2)}} \xi(\varphi,t).$$

The statement now follows by letting $\varepsilon \to 0$. ■

Let us now fix a $t > 0$ that satisfies (49). If $\sup_{\varphi\in\mathcal{F}} \xi(\varphi,t) > \xi(\varphi_0,t)$ then there must also exist a feasible simple function $\psi := \sum_{k=1}^m \frac{k}{m} 1_{A_k}$ with $\mathrm{supp}(\psi) \subseteq B(0;2)$ such that $\xi(\psi,t) > \xi(\varphi_0,t)$, because (by Lemma 2.1, item **(viii)**) for any φ the increasing sequence of functions $(\varphi_n)_n$ given by $\varphi_n = \sum_{k=1}^{2^n} \frac{k}{2^n} 1_{\{\frac{k}{2^n}\leq\varphi<\frac{k+1}{2^n}\}}$ satisfies $\lim_{n\to\infty} \xi(\varphi_n,t) = \xi(\varphi,t)$.

So let $\psi = \sum_{i=1}^m \frac{i}{m} 1_{A_i}$ be a feasible simple function with $\xi(\psi,t) > \xi(\varphi_0,t)$ and $\mathrm{supp}(\psi) \subseteq B(0;2)$. We may suppose wlog that the A_k are disjoint and m is even. For $1 \leq k \leq \frac{m}{2}$ let ψ_k be the function which is $\frac{1}{2}$ on $\bigcup_{i=k}^{m-k} A_i$ and 1 on $\bigcup_{i>m-k} A_i$. We can write

$$\psi = \frac{2}{m}\sum_{k=1}^{m/2} \psi_k,$$

because for $x \in A_i$ with $i \leq m/2$ we have $\frac{2}{m}\sum_{k=1}^{m/2} \psi_k(x) = i\frac{2}{m}\frac{1}{2} = \frac{i}{m}$, and if $x \in A_{m-i}$ with $i \leq m/2$ then $\frac{2}{m}\sum_{k=1}^{m/2} \varphi_k(x) = 1 - i\frac{2}{m}\frac{1}{2} = \frac{m-i}{m}$.

Let us now observe that ξ is convex in its first argument; that is, for any two nonnegative, bounded, measurable functions σ, τ and any $t > 0$ and $\lambda \in [0,1]$ we have

$$\xi(\lambda\sigma + (1-\lambda)\tau, t) \leq \lambda\xi(\sigma,t) + (1-\lambda)\xi(\tau,t).$$

This follows from parts **(ii)** and **(iii)** of Lemma 2.1. Because we have written ψ as a convex combination of the ψ_k, we must therefore have $\xi(\psi,t) \leq \xi(\psi_k,t)$ for some k.

Let us first assume that $\{\psi_k = 1\} = \bigcup_{l>m-k} A_k = \emptyset$. Since $\mathrm{supp}(\psi) \subseteq B(0;2)$ we must have that $\psi_k \leq \varphi'$, where φ' is the function which is $\frac{1}{2}$ on $B(0;3)$ and 0 elsewhere, and thus also $\xi(\psi,t) \leq \xi(\psi_k,t) \leq \xi(\varphi',t) = \frac{1}{2}\xi(1_{B(0;3)},t)$ (by choice of k and Lemma 2.1, items **(i)** and **(ii)**). But then (49) gives:

$$\xi(\psi,t) \leq \frac{1}{2}\xi(1_{B(0;3)},t) < \xi(\varphi_0,t),$$

a contradiction.

So we must have $\{\psi_k = 1\} \neq \emptyset$. Let us denote by $C := \mathrm{cl}(B)$ the closed unit ball. Notice that

$$\mathrm{diam}(\{\psi_k = 1\}) \leq 1,$$

$$\mathrm{supp}(\psi_k) \subseteq \bigcap_{x:\psi_k(x)=1} (x + C),$$

by feasibility of ψ (if $x \in \mathrm{supp}(\psi_k)$ and $\psi_k(y) = 1$ then $\psi(x) + \psi(y) > \frac{k}{m} + \frac{m-k}{m} = 1$). Bieberbach's inequality (Lemma 4.2) tells us that $\mathrm{vol}(\{\psi_k = 1\})$ cannot exceed $\frac{\mathrm{vol}(B)}{2^d}$, the volume of a ball of diameter 1. Hence there is a $0 \le \beta \le 1$ with $\mathrm{vol}(\{\psi_k = 1\}) = \mathrm{vol}(B(0; \frac{1-\beta}{2})) = (1-\beta)^d \frac{\mathrm{vol}(B)}{2^d}$. We will need another inequality, given by the following lemma.

Lemma 6.4 (K. Böröczky Jr., 2005) *Let $C \subseteq \mathbb{R}^d$ be a compact, convex set. Let $A \subseteq \mathbb{R}^d$ be measurable and let A' be a homothet (that is, a scaled copy) of $-C$ with* $\mathrm{vol}(A) = \mathrm{vol}(A')$. *Then*

$$\mathrm{vol}\left(\bigcap_{a \in A}(a + C)\right) \le \mathrm{vol}\left(\bigcap_{a \in A'}(a + C)\right).$$

With the kind permission of K. Böröczky Jr. we present a proof in appendix A, because such a proof is not readily available elsewhere.

It follows from Lemma 6.4 that we must have

$$\mathrm{vol}(\mathrm{supp}(\psi_k)) \le \mathrm{vol}\left(\bigcap_{x \in B(0;\frac{1-\beta}{2})}(x + C)\right) = \mathrm{vol}\left(B(0; \frac{1+\beta}{2})\right) = (1+\beta)^d \frac{\mathrm{vol}(B)}{2^d}.$$

For $0 \le \beta \le 1$ let φ_β be the function which is 1 on $B(0; \frac{1-\beta}{2})$ and $\frac{1}{2}$ on $B(0; \frac{1+\beta}{2}) \setminus B(0; \frac{1-\beta}{2})$. (This agrees with our earlier definition of φ_0.) We see that $\mathrm{vol}(\{\psi_k = 1\}) = \mathrm{vol}(\{\varphi_\beta = 1\})$ and $\mathrm{vol}(\{\psi_k = \frac{1}{2}\}) \le \mathrm{vol}(\{\varphi_\beta = \frac{1}{2}\})$. Thus we have $\int \psi_k 1_{\{\psi_k \ge a\}} \le \int \varphi_\beta 1_{\{\varphi_\beta \ge a\}}$ for all a, which gives $\xi(\psi_k, t) \le \xi(\varphi_\beta, t)$ by part **(vi)** of Lemma 2.1. We may conclude that if $\sup_{\varphi \in \mathcal{F}} \xi(\varphi, t) > \xi(\varphi_0, t)$ and (49) holds then also $\xi(\varphi_\beta, t) > \xi(\varphi_0, t)$ for some $0 < \beta \le 1$. We will show that this last statement is false. Set $\mu(\beta) := \xi(\varphi_\beta, t)$ for $0 \le \beta \le 1$.

Lemma 6.5

$$\max_{0 \le \beta \le 1} \mu(\beta) = \max\{\mu(0), \mu(1)\}.$$

Proof: Notice that for $0 \le \beta \le 1$

$$\mu(\beta) = \frac{\mathrm{vol}(B)}{2^d}(\frac{1}{2}((1+\beta)^d - (1-\beta)^d)e^{s/2} + (1-\beta)^d e^s),$$

where $s = s(\beta)$ solves

$$\frac{\mathrm{vol}(B)}{2^d}\left(((1+\beta)^d - (1-\beta)^d)H(e^{s/2}) + (1-\beta)^d H(e^s)\right) = \frac{1}{t}. \tag{54}$$

The function $\mu(\beta)$ is continuous on $[0, 1]$. Differentiating equation (54) wrt β we see that for $0 < \beta < 1$

$$\begin{aligned} 0 \; &= \; d((1+\beta)^{d-1} + (1-\beta)^{d-1})H(e^{s/2}) + ((1+\beta)^d - (1-\beta)^d))\tfrac{s}{4}e^{s/2}s' - d(1-\beta)^{d-1}H(e^s) \\ &\quad +(1-\beta)^d s e^s s'. \end{aligned}$$

(That s is differentiable wrt β can be justified using the implicit function theorem.) This gives

$$s'(\beta) = \frac{d(1-\beta)^{d-1}H(e^s) - d((1+\beta)^{d-1} + (1-\beta)^{d-1})H(e^{s/2})}{((1+\beta)^d - (1-\beta)^d)\frac{s}{4}e^{s/2} + (1-\beta)^d s e^s}.$$

Thus,

$$\begin{aligned}
\tfrac{\mathrm{vol}(B)}{2^d}\mu'(\beta) &= d((1+\beta)^{d-1} + (1-\beta)^{d-1})\tfrac{1}{2}e^{s/2} - d(1-\beta)^{d-1}e^s \\
&\quad + s'(((1+\beta)^d - (1-\beta)^d)\tfrac{1}{4}e^{s/2} + (1-\beta)^d e^s) \\
&= d((1+\beta)^{d-1} + (1-\beta)^{d-1})\tfrac{1}{2}e^{s/2} - d(1-\beta)^{d-1}e^s \\
&\quad + \tfrac{1}{s}(d(1-\beta)^{d-1}H(e^s) - d((1+\beta)^{d-1} + (1-\beta)^{d-1})H(e^{s/2})) \\
&= \tfrac{1}{s}\left[d((1+\beta)^{d-1} + (1-\beta)^{d-1})(e^{s/2}-1) - d(1-\beta)^{d-1}(e^s-1)\right].
\end{aligned}$$

Clearly $\mu'(\beta) > 0$ for β sufficiently close to 1, so that it suffices to show that (for any t) $\mu'(\beta) = 0$ for no more than one $\beta \in (0,1)$. Note that $\mu'(\beta) = 0$ if and only if

$$e^s - 1 = ((\frac{1+\beta}{1-\beta})^{d-1} + 1)(e^{s/2} - 1).$$

Writing $a := (\frac{1+\beta}{1-\beta})^{d-1} + 1$ and $x := e^{s/2}$ this translates into the quadratic $x^2 - ax + (a-1) = 0$, which has roots $1, a-1$. Now notice that $e^{s/2} = 1$ would give $s(\beta) = 0$, but this is never a solution of (54). So if $\mu'(\beta) = 0$ for some $0 < \beta < 1$ then we must have $s(\beta) = 2(d-1)\ln(\frac{1+\beta}{1-\beta})$. Notice that, as s cannot equal 0, this also shows that we must have $d \geq 2$ for $\mu'(\beta) = 0$ to hold. The curve $u(\beta) := 2(d-1)\ln(\frac{1+\beta}{1-\beta})$ has derivative

$$u'(\beta) = \frac{4(d-1)}{(1-\beta)(1+\beta)}.$$

On the other hand, for $0 < \beta < 1$

$$s'(\beta) < \frac{d(1-\beta)^{d-1}H(e^s)}{(1-\beta)^d s e^s} < \frac{d}{1-\beta}.$$

We find $s'(\beta) < \frac{d}{1-\beta} < 4(d-1)/(1+\beta)(1-\beta) = u'(\beta)$ for $0 < \beta < 1$ (recall $d \geq 2$ by a previous remark). We may conclude that the curves $u(\beta)$ and $s(\beta)$ meet in at most one point, as $u(\beta) - s(\beta)$ is strictly increasing on $(0,1)$. In other words, there is at most one $\beta \in (0,1)$ with $\mu'(\beta) = 0$ and the lemma follows. ■

Since we had chosen t so that (49) holds, parts **(iv)** and **(i)** of Lemma 2.1 tell us that:

$$\xi(\varphi_1, t) = \frac{1}{2}\xi(1_B, t) \leq \frac{1}{2}\xi(1_{B(0;3)}, t) < \xi(\varphi_0, t).$$

Thus

$$\max_{0 \leq \beta \leq 1} \mu(\beta) = \mu(0) = \xi(\varphi_0, t).$$

Observe that we have now achieved our aim in this subsection, namely to show that $t_0 > 0$. Thus we have proved the following lemma.

Lemma 6.6 *When the packing constant $\delta < 1$, we have $0 < t_0 < \infty$.*

6.2 When $\delta < 1$ the function $f_{\chi/\omega}(t)$ is strictly increasing for $t \geq t_0$

In this subsection we shall prove the result just stated in the heading. The proof uses the following lemma.

Lemma 6.7 *For each $t > 0$, either $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \frac{\text{vol}(B)}{2^d\delta}$ or the supremum is attained.*

Proof: Let us assume $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) > \frac{\text{vol}(B)}{2^d\delta}$ (as otherwise there is nothing to prove). Let us consider a sequence $\varphi_1, \varphi_2, \ldots \in \mathcal{F}$ such that

$$\lim_{n\to\infty} \xi(\varphi_n, t) = \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t), \tag{55}$$

and let us suppose (wlog) that

$$\lim_n \int_B \varphi_n \text{ exists and is as large as possible subject to (55)} \tag{56}$$

(recall that $B = B(0;1)$ is the unit ball). We will first exhibit a subsequence $\varphi_{n_1}, \varphi_{n_2}, \ldots$ of $(\varphi_n)_n$ and a function $\psi \in \mathcal{F}$ such that

$$\limsup_{k\to\infty} \varphi_{n_k}(x) \leq \psi(x) \text{ for all } x \in \mathbb{R}^d. \tag{57}$$

It will need further work to show that the function ψ achieves the supremum as required.

In order to construct ψ and the subsequence $(\varphi_{n_k})_k$, let $\mathcal{D}_k$ be the dissection $\{i + [0, 2^{-k})^d : i = (i_1, \ldots, i_d) \in 2^{-k}\mathbb{Z}^d\}$ of $\mathbb{R}^d$ into cubes of side 2^{-k} (observe that $\mathcal{D}_{k+1}$ refines $\mathcal{D}_k$). For $\sigma \in \mathcal{F}$ let us define the functions σ^k by setting:

$$\sigma^k(x) := \sup_{y\in C_{x,k}} \sigma(y),$$

where $C_{x,k}$ is the unique cube $C \in \mathcal{D}_k$ with $x \in C$. Let us now construct a nested sequence $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots$ of infinite subsets of $\{\varphi_1, \varphi_2, \ldots\}$ with the property that

$$|\sigma^k(x) - \tau^k(x)| \leq \frac{1}{k} \text{ for all } x \in [-k, k)^d \text{ and all } \sigma, \tau \in \mathcal{F}_k. \tag{58}$$

To see that this can be done, notice that the behaviour of σ^k on $[-k, k)^d$ is determined completely by $(\sigma^k(p_1), \ldots, \sigma^k(p_K))$ where $p_1, \ldots, p_K$ is some enumeration of $[-k, k)^d \cap 2^{-k}\mathbb{Z}^d$. Given $\mathcal{F}_{k-1}$ there must be intervals $I_1, \ldots, I_K \subseteq [0, 1]$ each of length $\frac{1}{k}$ such that the collection $\{\sigma \in \mathcal{F}_{k-1} : \sigma^k(p_i) \in I_i \text{ for all } 1 \leq i \leq K\}$ is infinite. So we can take $\mathcal{F}_k$ to be such an infinite collection. Let us now pick a subsequence $\varphi_{n_1}, \varphi_{n_2}, \ldots$ of $(\varphi_n)_n$ with $\varphi_{n_k} \in \mathcal{F}_k$ and let the function ψ be defined by:

$$\psi(x) := \lim_{k\to\infty} \varphi^k_{n_k}(x). \tag{59}$$

To see that this limit exists for all x, notice that $\varphi^l_{n_l}(x) \le \varphi^k_{n_l}(x) \le \varphi^k_{n_k}(x) + \frac{1}{k}$ for all $l \ge k > \|x\|_\infty$. Thus,

$$\begin{array}{rcl} \limsup_{k\to\infty} \varphi^k_{n_k}(x) & \le & \inf_{k>\|x\|_\infty} \varphi^k_{n_k}(x) + \frac{1}{k} \\ & = & \liminf_{k\to\infty} \varphi^k_{n_k}(x) + \frac{1}{k} \\ & = & \liminf_{k\to\infty} \varphi^k_{n_k}(x). \end{array}$$

We now claim that ψ and the sequence $(\varphi_{n_k})_k$ are as required (that is, $\psi \in \mathcal{F}$ and (57) holds). To see that (57) holds, notice that $\sup_{l\ge k} \varphi_{n_l}(x) \le \varphi^k_{n_k}(x) + \frac{1}{k}$ for any x and any $k > \|x\|_\infty$, so that

$$\limsup_{k\to\infty} \varphi_{n_k}(x) \le \lim_{k\to\infty} \varphi^k_{n_k}(x) = \psi(x). \tag{60}$$

To see that $\psi \in \mathcal{F}$, let $S = \{s_1, \dots, s_p\} \in \mathcal{S}$ be finite (observe it suffices to show $\sum_{x\in S} \psi(x) \le 1$ for all finite $S \in \mathcal{S}$). Since $\|s_i - s_j\| > 1$ for all $i \ne j$, there is a k_0 such that $\|s_i - s_j\| > 1 + 2^{-k_0}\rho$ for all $i \ne j$ where $\rho := \mathrm{diam}([0,1]^d)$. Thus if $k \ge k_0$ then

$$\varphi^k_{n_k}(s_1) + \dots + \varphi^k_{n_k}(s_p) \le 1,$$

and hence the same must hold for ψ.

Also notice that the dominated convergence theorem (using $\psi, \varphi_{n_k} \le 1$) gives that

$$\int_B \psi(x)\mathrm{d}x = \lim_{k\to\infty} \int_B \varphi^k_{n_k}(x)\mathrm{d}x \ge \lim_{n\to\infty} \int_B \varphi_n(x)\mathrm{d}x, \tag{61}$$

using (60) for the second equation. Furthermore, for any fixed $R > 0$ we have that

$$\lim_{n\to\infty} \xi(\varphi^k_{n_k} 1_{B(0;R)}, t) = \xi(\psi 1_{B(0,R)}, t) \le \xi(\psi, t). \tag{62}$$

Here we have used parts **(i)** and **(vii)** of Lemma 2.1. Hence, there also is a sequence $(R_k)_k$ with R_k tending to infinity and

$$\limsup_{k\to\infty} \xi(\varphi^k_{n_k} 1_{B(0;R_k)}, t) \le \xi(\psi, t). \tag{63}$$

To see this, notice that by (62) there exist $k_1 < k_2 < \dots$ such that $\xi(\varphi^k_{n_k} 1_{B(0;m)}, t) \le \xi(\psi, t) + \frac{1}{m}$ for all $k \ge k_m$. Thus, we may put $R_k := \max\{m : k_m \le k\}$.

Let us put

$$\psi_{k,i} := \varphi_{n_k} 1_{B(0;R_k)}, \quad \psi_{k,o} := \varphi_{n_k} 1_{\mathbb{R}^d \setminus B(0;R_k+1)}, \quad \psi_k := \psi_{k,i} + \psi_{k,o}. \tag{64}$$

We may assume wlog that R_k has been chosen in such a way that $\xi(\psi_k, t) = (1 + o(1))\xi(\varphi_{n_k}, t)$. To see this note that for $s = s(\varphi_{n_k}, t)$ there is an $\frac{R_k}{2} \le R' \le R_k$ s.t.

$$\int \varphi_{n_k} e^{s\varphi_{n_k}} 1_{B(0;R'+1)\setminus B(0;R')} \le \frac{1}{\lfloor \frac{R_k}{2} \rfloor} \int \varphi_{n_k} e^{s\varphi_{n_k}}.$$

If we take such an R' and set $\psi'_k := \varphi_{n_k} 1_{\mathbb{R}^d \setminus B(0;R'+1) \cup B(0;R')}$ then $s(\psi'_k, t) \geq s(\varphi_{n_k}, t)$ (by the definition of s, as $\psi'_k \leq \varphi_{n_k}$) so that $\xi(\psi'_k, t) \geq (1 - \frac{1}{\lfloor \frac{R_k}{2} \rfloor})\xi(\varphi_{n_k}, t)$.

Observe that by our choice of R_k

$$\lim_{k\to\infty} \xi(\psi_k, t) = \lim_{k\to\infty} \xi(\varphi_{n_k}, t) = \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t), \tag{65}$$

and (since $R_k \geq 1$ for k sufficiently large)

$$\lim_{k\to\infty} \int_B \psi_k = \lim_{k\to\infty} \int_B \varphi_{n_k} = \lim_{n\to\infty} \int_B \varphi_n. \tag{66}$$

Let us define $\lambda(\varphi) := \sup_{S\in\mathcal{S}} \sum_{x\in S} \varphi(x)$. Since $\psi_k \leq \varphi_{n_k} \in \mathcal{F}$ we have

$$\lambda(\psi_k) = \lambda(\psi_{k,i}) + \lambda(\psi_{k,o}) \leq 1. \tag{67}$$

For convenience let us write $\lambda_k := \lambda(\psi_{k,o})$. First let us suppose that $\lambda_k \to 0$ as $k \to \infty$. Notice that $\frac{1}{\lambda_k}\psi_{k,o} \in \mathcal{F}$, which implies

$$\xi(\psi_{k,o}, t) \leq \lambda_k \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = o(1),$$

using part **(ii)** of Lemma 2.1. Observe that

$$\xi(\psi_{k,i}, t) \leq \xi(\psi_k, t) \leq \xi(\psi_{k,i}, t) + \xi(\psi_{k,o}, t)$$

by parts **(i)** and **(iii)** of Lemma 2.1. Thus, $\xi(\psi_k, t) = \xi(\psi_{k,i}, t) + o(1)$. Using (65) and (63)

$$\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \lim_{k\to\infty} \xi(\psi_k, t) \leq \xi(\psi, t),$$

so that the lemma follows in the case when $\lambda_k \to 0$.

In the remaining part of the proof we shall show that in fact we must have $\lambda_k \to 0$. Let us assume not, and that $\limsup \lambda_k > 0$. We may assume for convenience that $\lim \lambda_k = \lambda > 0$ (by considering a subsequence if necessary). We next establish the following claim.

Claim

$$\mathrm{vol}(\{\psi_{k,o} \geq \varepsilon\}) \to 0 \quad \text{ for all } \varepsilon > 0. \tag{68}$$

Proof of (68)**:** Let us construct a new sequence of functions ψ'_k as follows. For each k pick an $x_k \in \mathbb{R}^d \setminus B(0; R_k + 1)$ that maximises $\int_{B(x_k;1)} \psi_{k,o}$.

To see that such an x_k exists, let us write $I(x) := \int_{B(x;1)} \psi_{k,o}$. Notice that I is continuous ($\psi_{k,o} \leq 1$ so that $|I(x) - I(y)| \leq \mathrm{vol}(B(x;1) \setminus B(y;1))$). Let us suppose that $c := \sup_{x\in\mathbb{R}^d\setminus B(0;R_k+1)} I(x) > 0$, for otherwise there is nothing to prove as any $x \in \mathbb{R}^d \setminus B(0; R_k+1)$ will do. We first claim that the set $\{x \in \mathbb{R}^d \setminus B(0; R_k+1) : I(x) > \frac{c}{2}\}$ can be covered by at most $\lfloor \frac{2\,\mathrm{vol}(B)}{c} \rfloor$ balls of radius two. This is because if $I(x_1), \ldots, I(x_k) > \frac{c}{2}$ there must exist $y_i \in B(x_i; 1)$ for $1 \leq i \leq k$ with $\psi_{k,o}(y_i) > \frac{c}{2\,\mathrm{vol}(B)}$. By feasibility of $\psi_{k,o}$

we must have either $k < \frac{2\,\mathrm{vol}(B)}{c}$ or $\|y_i - y_j\| \le 1$ for some $i \ne j$. Thus, the y_i can be covered by at most $\frac{2\,\mathrm{vol}(B)}{c}$ balls of radius one, and hence the x_i can be covered by at most $\frac{2\,\mathrm{vol}(B)}{c}$ balls of radius two, as claimed. As I is continuous and we can restrict ourselves to a compact subset of $\mathbb{R}^d \setminus B(0; R_k+1)$ we see that the supremum $c = \sup_{x\in\mathbb{R}^d\setminus B(0;R_k+1)} I(x)$ is indeed attained by some point $x_k \in \mathbb{R}^d \setminus B(0; R_k + 1)$.

Now let $\psi'_k := \psi_{k,i} + \psi_{k,o} \circ T_k$ where $T_k : y \mapsto y + x_k$ is the translation that sends 0 to x_k. By (67) we have $\psi'_k \in \mathcal{F}$. Notice that

$$\int \psi'_k 1_{\{\psi'_k \ge a\}} \ge \int \psi_k 1_{\{\psi_k \ge a\}}$$

for all a, because $\{\psi'_k \ge a\} \supseteq \{\psi_{k,i} \ge a\} \cup T_k^{-1}[\{\psi_{k,o} \ge a\}]$ so that

$$\begin{aligned}
\int \psi'_k 1_{\{\psi'_k \ge a\}} &\ge \int \psi_{k,i} 1_{\{\psi_{k,i}\ge a\}} + \int (\psi_{k,o} \circ T_k) 1_{T_k^{-1}[\{\psi_{k,o}\ge a\}]} \\
&= \int \psi_{k,i} 1_{\{\psi_{k,i}\ge a\}} + \int \psi_{k,o} 1_{\{\psi_{k,o}\ge a\}} \\
&= \int \psi_k 1_{\{\psi_k \ge a\}}.
\end{aligned}$$

Part **(vi)** of Lemma 2.1 therefore gives that

$$\xi(\psi'_k, t) \ge \xi(\psi_k, t) = (1 + o(1))\xi(\varphi_{n_k}, t).$$

Thus $\xi(\psi'_k, t) \to \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t)$ as $k \to \infty$, and we saw earlier that $\psi'_k \in \mathcal{F}$. We therefore must have $\int_{B(x_k;1)} \psi_{k,o} \to 0$; for otherwise, using (66), we have $\limsup_k \int_B \psi'_k > \lim_k \int_B \psi_k = \lim_n \int_B \varphi_n$, and (a subsequence of) the ψ'_k would contradict (56). Now suppose that for some $\varepsilon > 0$ we have $\limsup_k \mathrm{vol}(\{\psi_{k,o} \ge \varepsilon\}) = c > 0$. Because $\psi_{k,o} \in \mathcal{F}$ we can cover $\{\psi_{k,o} \ge \varepsilon\}$ by at most $\lfloor \frac{1}{\varepsilon} \rfloor$ balls of radius 1. But this gives $\limsup_k \int_{B(x_k;1)} \psi_{k,o}(x)\mathrm{d}x \ge c\varepsilon$ and we know this cannot happen. The claim (68) follows. ■

Recall that $\sigma_k := \frac{1}{\lambda_k}\psi_{k,o} \in \mathcal{F}$. Because $\lim_k \lambda_k = \lambda > 0$ the previous also gives $\lim_k \mathrm{vol}(\{\sigma_k \ge \varepsilon\}) = 0$ for all $\varepsilon > 0$. Let us fix $\varepsilon > 0$ for now and let $V_\varepsilon, W_{k,\varepsilon} \subseteq \mathbb{R}^d$ be disjoint (measurable) sets with $\mathrm{vol}(V_\varepsilon) = \frac{\mathrm{vol}(B)}{\varepsilon 2^d \delta}$ and $\mathrm{vol}(W_{k,\varepsilon}) = \mathrm{vol}(\{\sigma_k \ge \varepsilon\})$. Let us set $\tau_k := \varepsilon 1_{V_\varepsilon} + 1_{W_{k,\varepsilon}}$. Then $\int \sigma_k 1_{\{\sigma_k\ge a\}} \le \int \tau_k 1_{\{\tau_k \ge a\}}$ for all a (using $\sigma_k \le 1$ and Lemma 5.3), so that parts **(vi)**, **(ii)** and **(iii)** of Lemma 2.1 give:

$$\begin{aligned}
\xi(\sigma_k, t) \le \xi(\tau_k, t) &\le \varepsilon\xi(1_{V_\varepsilon}, t) + \xi(1_{W_{k,\varepsilon}}, t) \\
&= \varepsilon c(\mathrm{vol}(V_\varepsilon), t) + c(\mathrm{vol}(W_{k,\varepsilon}), t).
\end{aligned}$$

Now $x = c(w, t)w$ solves $H(x) = 1/(wt)$, so $x \to 1$ as $w \to 0$; and thus $c(w, t) \sim w$ as $w \to 0$. Hence $\varepsilon c(\mathrm{vol}(V_\varepsilon), t) \sim \varepsilon\,\mathrm{vol}(V_\varepsilon) = \frac{\mathrm{vol}(B)}{2^d\delta}$ as $\varepsilon \to 0$; and for any fixed $\varepsilon > 0$ $c(\mathrm{vol}(W_{k,\varepsilon}), t) \to 0$ as $k \to \infty$.

It follows that $\limsup_{k\to\infty} \xi(\sigma_k, t) \le \frac{\mathrm{vol}(B)}{2^d\delta}$. Since $\sigma'_k := \frac{1}{1-\lambda_k}\psi_{k,i} \in \mathcal{F}$ by (67), we thus have

$$\lim_{k\to\infty} \xi(\psi_k, t) = \lim_{k\to\infty} \xi(\lambda_k\sigma_k + (1-\lambda_k)\sigma'_k, t) \le \lambda\frac{\mathrm{vol}(B)}{2^d\delta} + (1-\lambda)\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) < \sup_{\varphi\in\mathcal{F}} \xi(\varphi, t),$$

using parts **(i)** and **(ii)** of Lemma 2.1. But this contradicts equation (65): so we must have $\lambda_k \to 0$, completing the proof of Lemma 6.7. ■

Lemma 6.8 *Assume that $\delta < 1$ (so $0 < t_0 < \infty$). Then the function $f_{\chi/\omega}(t)$ is continuous and strictly increasing for $t_0 \le t < \infty$.*

Proof: That $f_{\chi/\omega}(t)$ is continuous follows immediately from parts **(iii)** of Theorems 1.1 and 1.2, which we have already established in sections 4 and 5. Let us observe that, by definition (48) of t_0, it suffices to show that whenever $t > 0$ is such that $f_{\chi/\omega}(t) > 1$ then $f_{\chi/\omega}(t') > f_{\chi/\omega}(t)$ for all $t' > t$.

Consider a $t > t_0$ with $f_{\chi/\omega}(t) > 1$. First suppose that $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) = \frac{\mathrm{vol}(B)}{2^d\delta}$. Notice that the lower bound in Lemma 5.4 then shows that also $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t') = \frac{\mathrm{vol}(B)}{2^d\delta}$ for all $t' \ge t$, so that in this case $f_{\chi/\omega}(t') > f_{\chi/\omega}(t)$ for all $t' > t$ as $\xi(\varphi_0, t)$ is strictly decreasing in t.

So we may assume that $\sup_{\varphi\in\mathcal{F}} \xi(\varphi, t) > \frac{\mathrm{vol}(B)}{2^d\delta}$. By Lemma 6.7 there is a $\varphi \in \mathcal{F}$ with $0 < \int \varphi < \infty$ such that the supremum equals $\xi(\varphi, t)$. Next, we claim that would suffice to prove that for any $\lambda > 1$ there is at most one $t' > 0$ that solves the equation $\xi(\varphi, t') = \lambda\xi(\varphi_0, t')$. This is because, since $\xi(\varphi, t_0) \le \xi(\varphi_0, t_0)$ and $t' \mapsto \xi(\varphi, t')/\xi(\varphi_0, t')$ is continuous in t', we would then also get that

$$f_{\chi/\omega}(t') \ge \frac{\xi(\varphi, t')}{\xi(\varphi_0, t')} > \frac{\xi(\varphi, t)}{\xi(\varphi_0, t)} = f_{\chi/\omega}(t),$$

for all $t' > t$.

Set $\psi := \lambda\varphi_0$ with $\lambda > 1$. By Lemma 2.1, part **(ii)**, $\xi(\psi, t) = \lambda\xi(\varphi_0, t)$. So to prove the lemma, it suffices to show that the system of equations

$$\int_{\mathbb{R}^d} \varphi(x) e^{w\varphi(x)} \mathrm{d}x = \int_{\mathbb{R}^d} \psi(x) e^{s\psi(x)} \mathrm{d}x, \tag{69}$$

$$\int_{\mathbb{R}^d} H(e^{w\varphi(x)}) \mathrm{d}x = \int_{\mathbb{R}^d} H(e^{s\psi(x)}) \mathrm{d}x \tag{70}$$

has at most one solution (w, s) with $w, s > 0$. For $s \ge 0$ let $v(s)$ be the unique solution of (69) and let $u(s)$ be the unique non-negative solution of (70). Let us write

$$F(s) := v(s) - u(s).$$

Our goal for the remainder of the proof will be to show that there is at most one solution $s > 0$ of $F(s) = 0$, which implies that there is at most one solution (w, s) with $w, s > 0$ of the system given by (69) and (70) and thus also implies the lemma.

Differentiating both sides of equation (69) wrt s we get

$$\int_{\mathbb{R}^d} v'(s)\varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x = \int_{\mathbb{R}^d} \psi^2(x) e^{s\psi(x)} \mathrm{d}x,$$

where we have swapped integration wrt x and differentiation wrt s. (This can be justified using the fundamental theorem of calculus and Fubini's theorem for nonnegative functions; and that v is differentiable can be justified using the implicit function theorem – for more details see the footnotes in section 2.) This gives

$$v'(s) = \frac{\int_{\mathbb{R}^d} \psi^2(x) e^{s\psi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x}. \tag{71}$$

Similarly, differentiating (70) wrt s we get that for $s > 0$:

$$u'(s) = \frac{s \int_{\mathbb{R}^d} \psi^2(x) e^{s\psi(x)} \mathrm{d}x}{u(s) \int_{\mathbb{R}^d} \varphi^2(x) e^{u(s)\varphi(x)} \mathrm{d}x}. \tag{72}$$

Observe that from (71) we get

$$v'(s) = \lambda \frac{\int_{\mathbb{R}^d} \psi(x) e^{s\psi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x} \geq \lambda \frac{\int_{\mathbb{R}^d} \psi(x) e^{s\psi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi(x) e^{v(s)\varphi(x)} \mathrm{d}x} = \lambda, \tag{73}$$

where we have used the specific form of ψ as a constant times an indicator function, the fact that $\varphi^2(x) \leq \varphi(x)$ (as $0 \leq \varphi(x) \leq 1$ for all x) and the fact that $v(s)$ solves (69).
From (71) and (72) we see that:

$$\text{If } u(s) = v(s) \text{ for some } s > 0 \text{ then } u'(s) = \frac{s}{v(s)} v'(s). \tag{74}$$

Let us first suppose that $v(0) \geq 0$. Since $v'(s) \geq \lambda > 1$ for all $s > 0$, it follows that $v(s) > s$ for all $s > 0$. Thus, from (74) we see that whenever $v(s) = u(s)$ we have $u'(s) < v'(s)$. In other words, at every zero of F we have $F'(s) = v'(s) - u'(s) > 0$. This shows F can have at most one zero, as required.

It remains to consider the case when $v(0) < 0$. We may assume that there is a solution $s > 0$ to $v(s) = u(s)$, and since F is continuous there is a least such solution s_1. Now $u(0) = 0$ so $F(0) < 0$, and thus $F(s) < 0$ for each $0 \leq s < s_1$. Hence $F'(s_1) = v'(s_1) - u'(s_1) \geq 0$. From (74) we now see that $v(s_1) \geq s_1$, so that using (73) we have $v(s) > s$ for all $s > s_1$. Reasoning as before this gives that:

$$\text{If } F(s_2) = 0 \text{ for some } s_2 > s_1 \text{ then } F'(s_2) > 0. \tag{75}$$

Hence, if $F(s) > 0$ for some $s > s_1$ then $F(s') > 0$ for all $s' \geq s$: for if not then there is a least $s' > s$ such that $F(s') = 0$. Such an s' must then satisfy $F'(s') \leq 0$, contradicting (75).

Now suppose there is an $s_2 > s_1$ such that $F(s_2) = 0$. By the above we must then have $F(s) \leq 0$ for all $s_1 < s < s_2$. Hence, for $s_1 < s < s_2$ we have

$$\frac{v'(s)}{u'(s)} = \frac{u(s)}{s} \frac{\int_{\mathbb{R}^d} \varphi^2(x) e^{u(s)\varphi(x)} \mathrm{d}x}{\int_{\mathbb{R}^d} \varphi^2(x) e^{v(s)\varphi(x)} \mathrm{d}x} \geq \frac{v(s)}{s} > 1,$$

(using that $u(s) \geq v(s)$ and $v(s) > s$). Thus $F'(s) > 0$ for all $s_1 < s < s_2$. But this implies $F(s_2) > F(s_1) = 0$, contradicting the choice of s_2.

It follows that s_1 is the only zero of F, which concludes the proof of the lemma. ■

Lemmas 6.6 and 6.8 give Theorem 1.3.

7 Remaining proofs

There are still some loose ends. Here we will finish the proof of Theorems 1.1, 1.2, 1.4 and 1.7 and Proposition 1.6.

7.1 Proof of parts (ii) of Theorems 1.1 and 1.2

The following Lemma will immediately imply parts **(ii)** of Theorems 1.1 and 1.2; and it is also convenient for the proof of Theorem 1.4 later on. Recall that the notation A_n a.a.a.s. means that $\mathbb{P}(A_n$ holds for all but finitely many $n) = 1$.

Lemma 7.1 *For every $\varepsilon > 0$, there exists a $\beta = \beta(\sigma, \varepsilon)$ such that if $n^{-\beta} \leq nr^d \leq \beta \ln n$ for all sufficiently large n, then*

$$(1-\varepsilon)k(n) \leq \omega(G_n), \chi(G_n) \leq (1+\varepsilon)k(n) \text{ a.a.a.s.},$$

where $k(n) := \ln n / \ln\left(\frac{\ln n}{nr^d}\right)$.

Proof: Set $W_1 := B(0; \frac{1}{2})$ and $W_2 := \mathrm{cl}(B)$, and observe that

$$M_{W_1} \leq \omega(G_n) \leq \chi(G_n) \leq \Delta(G_n) + 1 \leq M_{W_2}, \tag{76}$$

where Δ denotes the maximum degree. Now set $\beta := \min(\beta(W_1, \sigma, \varepsilon), \beta(W_2, \sigma, \varepsilon))$ with $\beta(W, \sigma, \varepsilon)$ as in Lemma 3.9. The statement immediately follows from Lemma 3.9 together with (76). ■

7.2 Proof of parts (i) of Theorems 1.1 and 1.2 and of Proposition 1.6

Our plan is to use Lemma 3.8 on the generalised scan statistic. To make use of this lemma, we 'split r into parallel sequences'. Let $K \in \mathbb{N}$ be such that $\frac{1}{K} < \alpha$. For $k = 0, 1, \ldots, K$ set $a_k := \frac{1}{k+\frac{1}{2}}$ and set:

$$r_k(n) := \begin{cases} r(n) & \text{if } n^{-a_{k-1}} \leq nr^d < n^{-a_k}, \\ n^{-(a_{k-1}+1)/d} & \text{otherwise} \end{cases},$$

for $1 \leq k \leq K$ and for $k = 0$ set:

$$r_0(n) := \begin{cases} r(n) & \text{if } nr^d < n^{-a_0}, \\ n^{-4/d} & \text{otherwise} \end{cases}.$$

Observe that $n^{-\frac{1}{k-\frac{1}{2}}} \leq nr_k^d < n^{-\frac{1}{k+\frac{1}{2}}}$ for $k = 1, \ldots, K$ and $nr_0^d < n^{-2}$. Hence

$$r(n) = r_k(n) \quad \text{if and only if} \quad k = \left\lfloor \left|\tfrac{\ln n}{\ln(nr^d)}\right| + \tfrac{1}{2} \right\rfloor. \tag{77}$$

Let us now put $G_n^{(k)} := G(X_1, \ldots, X_n; r_k(n))$ for $1 \leq k \leq K$. Because G_n coincides with some $G_n^{(k)}$ for each n, and the intersection of finitely many events of probability one has probability one itself, it suffices to show that $\chi(G_n^{(k)}) = \omega(G_n^{(k)}) \in \{k, k+1\}$ a.a.a.s. for each k separately.

Let us thus suppose that there is a fixed $0 \leq k \leq K$ such that $r(n) = r_k(n)$ for all n. Set $W_1 := B(0; \frac{1}{2})$ and $W_2 := B(0; 100)$. Then we again have

$$M_{W_1} \leq \omega(G_n) \leq \chi(G_n) \leq \Delta(G_n) + 1 \leq M_{W_2}. \tag{78}$$

(The reason for choosing radius larger than 1 will become clear shortly.) Now notice that part **(i)** of Lemma 3.8 shows that a.a.a.s.:

$$M_{W_2} \leq k + 1. \tag{79}$$

If $k \in \{0, 1\}$ then $M_{W_1} \geq k$ holds trivially, and for $k \geq 2$ we can apply part **(ii)** of Lemma 3.8 to show that a.a.a.s.:

$$M_{W_1} \geq k. \tag{80}$$

Putting (78), (79) and (80) together, we see that we have just proved parts **(i)** of Theorems 1.1 and 1.2.

To finish the proof we will derive (deterministically) that if (79) and (80) hold then $\chi(G_n) = \omega(G_n)$ must also hold. So let us assume that (79) and (80) hold. First note that $\Delta(G_n) \in \{\omega(G_n) - 1, \omega(G_n)\}$. If $\Delta(G_n) = \omega(G_n) - 1$ then we are done (since always $\chi(G) \leq \Delta(G) + 1$), so let us suppose that $\Delta(G_n) = \omega(G_n)$. In this case Brooks' Theorem (see for example van Lint and Wilson [20]) tells us that $\chi(G_n) = \omega(G_n)$ unless $\omega(G_n) = 2$ and G_n contains an odd cycle of length at least 5. Let us therefore assume that $\omega(G_n) = 2$. Then we must have $k \leq 2$ and hence $M_{W_2} \leq 3$. But now each component of G_n must have at most 3 vertices: to see this note that if the subgraph induced by $X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}$ is connected then all four points are contained in a ball of radius $< 100r$. Hence there are no odd cycles of length at least 5 and so $\chi(G_n) = 2$ as required.

7.3 Proof of Theorem 1.3

Let r be any sequence of positive numbers that tends to 0, and let $\varepsilon > 0$ be arbitrary, but fixed. Our aim will be to prove

$$1 - \varepsilon \leq \frac{\chi(G_n)}{f_{\chi/\omega}(\frac{\sigma nr^d}{\ln n})\omega(G_n)} < 1 + \varepsilon \text{ a.a.a.s.}, \tag{81}$$

which clearly implies the theorem.

Set $\varepsilon' := \frac{1+\varepsilon}{1-\varepsilon} - 1$, and let $0 < \beta \leq \beta(\sigma, \varepsilon')$ where $\beta(\sigma, \varepsilon')$ is as in Lemma 7.1 above. We can assume without loss of generality that $\beta < t_0/\sigma$ with t_0 as in Theorem 1.3 if $\delta < 1$. (If $\delta = 1$ then we do not need any additional assumption.) We will apply the same trick we used in the previous section. Set $r_1(n) := \min(r(n), n^{-(\beta+1)/d})$,

$$r_2(n) := \begin{cases} r(n) & \text{if } n^{-\beta} < nr^d < \beta \ln n, \\ \left(\frac{\beta \ln n}{n}\right)^{\frac{1}{d}} & \text{otherwise.} \end{cases},$$

and $r_3(n) := \max(\left(\frac{\beta \ln n}{n}\right)^{\frac{1}{d}}, r(n))$. Now set $G_n^{(k)} := G(X_1, \ldots, X_n; r_k(n))$ for $k = 1, 2, 3$. It again suffices to prove (81) for each $G_n^{(k)}$ separately.

Notice that $\frac{\sigma n r_1^d}{\ln n} \to 0$ as $n \to \infty$, so that $f_{\chi/\omega}(\frac{\sigma n r_1^d}{\ln n}) \to 1$. The statement (81) for $G_n^{(1)}$ now follows immediately from Proposition 1.6.

Since $\liminf_n \frac{\sigma n r_3^d}{\ln n} > 0$, the statement (81) for $G_n^{(3)}$ follows immediately from Theorems 5.11 and 4.1.

Finally, notice that $\frac{\sigma n r_2^d}{\ln n} < t_0$ for all n, so that in fact $f_{\chi/\omega}(\frac{\sigma n r_2^d}{\ln n}) = 1$ for all n. Applying Lemma 7.1 we see that

$$1 \leq \frac{\chi(G_n^{(2)})}{\omega(G_n^{(2)})} < \frac{1+\varepsilon'}{1-\varepsilon'} = 1 + \varepsilon \text{ a.a.a.s.}$$

This concludes the proof of Theorem 1.4

7.4 Proof of Theorem 1.7

When $\delta = 1$ then the theorem follows directly from $\omega(G_n) \leq \chi_f(G_n) \leq \chi(G_n)$ together with part **(i)** of Theorem 1.3 and Theorem 1.4.

Let us thus suppose that $\delta < 1$. We can again "split into subsequences" as in sections 7.2 and 7.3, to see that it suffices to prove the theorem for the case when $\sigma n r^d \leq t_0 \ln n$ and the case when $\sigma n r^d > t_0 \ln n$.

If $\sigma n r^d \leq t_0 \ln n$ then $f_{\chi/\omega}(\frac{\sigma n r^d}{\ln n}) = 1$ so that the argument for the $\delta = 1$ case applies.

Finally, if $\sigma n r^d > t_0 \ln n$ then in particular $\liminf \frac{n r^d}{\ln n} > 0$; and now we may use (47) to complete the proof.

8 Concluding remarks

In this paper we have proved a number of almost sure convergence results on the chromatic number of the random geometric graph and we have investigated its relation to the clique number. Amongst other things we have set out to describe the "phase change" regime when $nr^d = \Theta(\ln n)$. An important shift in the behaviour of the chromatic number occurs in this range of r (except in the less interesting case when the packing constant $\delta = 1$). We have seen that (except when $\delta = 1$) there exists a finite positive constant t_0 such that if

$\sigma n r^d \leq t_0 \ln n$ then the chromatic number and the clique number of the random geometric graph are essentially equal in the sense that

$$\frac{\chi(G_n)}{\omega(G_n)} \to 1 \text{ a.s. ;}$$

and if on the other hand $\sigma n r^d \geq (t_0 + \varepsilon) \ln n$ for some fixed (but arbitrarily small) $\varepsilon > 0$ then the lim inf of this ratio is bounded away from 1 almost surely. Moreover, if $nr^d \gg \ln n$ then

$$\frac{\chi(G_n)}{\omega(G_n)} \to \frac{1}{\delta} \text{ a.s.}$$

where δ is the packing constant.

We have also given expressions for the almost sure limit $f_\chi(t)$ of $\frac{\chi(G_n)}{\sigma n r^d}$ and the almost sure limit $f_{\chi/\omega}(t)$ of $\frac{\chi(G_n)}{\omega(G_n)}$ if $\sigma n r^d \sim t \ln n$ for some $t > 0$. An interesting observation is that t_0 and the limiting functions $f_\omega(t)$, $f_\chi(t)$ and $f_{\chi/\omega}(t)$ do not depend on the choice of probability measure ν, and that the only feature of the probability measure that plays any role in the proofs and results in this paper is the maximum density σ.

It should be mentioned that considering the ratio $\frac{\chi(G_n)}{\omega(G_n)}$, apart from the fact that it provides an easy to state summary of the results, can also be motivated by the fact that while colouring unit disk graphs (non-random geometric graphs when $d = 2$ and $\|.\|$ is the Euclidean norm) is NP-hard (Clark et al [3], Gräf et al [5]), their clique number may be found in polynomial time [3], unlike finding the clique number in general graphs. In fact the clique number of a unit disk graph may be found in polynomial time even if an embedding (that is, an explicit representation with points on the plane) is not given (Raghavan and Spinrad [17]). Thus, the results given here suggest that even though finding the chromatic number of a unit disk graph G is NP-hard, for graphs that are not very sparse the polynomial approximation of finding the clique number and multiplying this by $\frac{1}{\delta}$ might work quite well in practice. Note that for the Euclidean norm in the plane $\frac{1}{\delta} = \frac{2\sqrt{3}}{\pi} \approx 1.103$: also in this case always $\chi(G)/\omega(G) < 3$ (Peeters [15]).

It is instructive to consider for comparison the ratio of chromatic number to clique number in the Erdős-Rényi model, with expected degree similar to the values we have been investigating. Let us consider $p = p(n)$ such that $np \to \infty$ with $np = o(n^{\frac{1}{3}})$ as $n \to \infty$. Then $\omega(G(n,p)) = 3$ whp (see for example Bollobás [?] Theorem 4.13), and $\chi(G(n,p)) \sim \frac{np}{2 \ln np}$ whp (Łuczak [9], or see [?] Theorem 11.29); and thus whp

$$\frac{\chi(G(n,p))}{\omega(G(n,p))} \sim \frac{np}{6 \ln np} \to \infty \text{ whp .}$$

Also, when does $\chi(G(n,p)) = \omega(G(n,p))$ whp? This property holds if $np \to 0$ as $n \to \infty$, since $G(n,p)$ is then a forest whp; and conversely. if the property holds (and p is bounded below one) then $np \to 0$. Thus, the results on the chromatic number given here, and in the earlier work of the first author [12] and Penrose [16], highlight a dramatic difference between the Erdős-Rényi model on the one hand and the random geometric model on the other hand.

Although we have presented substantial progress on the current state of knowledge on the chromatic number of random geometric graphs in this paper, several questions remain. Our proofs for instance do not yield an explicit expression for t_0 (when $\delta < 1$) and it would certainly be of interest to find such an expression or to give some (numerical) procedure to determine it, in particular for the euclidean norm in $\mathbb{R}^2$. More generally, it is far from trivial to extract information from the expressions for $f_\chi(t)$ and $f_{\chi/\omega}(t)$. In particular we would be interested to know whether f_χ is differentiable. In particular: is f_χ differentiable at t_0? Also Lemma 6.7 suggests the question: Is $f_\chi(t) > \frac{\mathrm{vol}(B)}{2^d\delta}$ for all $0 < t < \infty$?

Another natural question concerns the extent to which the chromatic number is 'local' or 'global'. Define the random variable R_n to be the infimum of the values $R > 0$ such that G_n contains a subgraph H_n which is induced by the points in some ball of radius R and which satisfies $\chi(H_n) = \chi(G_n)$. In the very sparse case when $nr^d \leq n^{-\alpha}$ for some fixed $\alpha > 0$, the proof of Proposition 1.6 shows that $R_n \leq \varepsilon r$ a.a.a.s. for any fixed $\varepsilon > 0$, so the behaviour is 'very local'. Now suppose that $\sigma n r^d \sim t \ln n$ for some $t > 0$. Then by Lemmas 6.1 and 6.2, for t sufficiently small we have $R \leq 2r$ a.a.a.s. so still the behaviour is local. But what happens for large t?

A question that has not been addressed in this paper (except in the "very sparse" case when nr^d is bounded by a negative power of n) is the probability distribution of $\chi(G_n)$. In a recent paper by the second author [13] it was shown that whenever $nr^d \ll \ln n$ then $\chi(G_n)$ is two-point concentrated, in the sense that

$$\mathbb{P}(\chi(G_n) \in \{k(n), k(n)+1\}) \to 1,$$

as $n \to \infty$ for some sequence $k(n)$. Analogous results were also shown to hold for the clique number, the maximum degree and the degeneracy of G_n. For other choices of r the distributions of these random variables are not known. However, it is possible to extend an argument in Penrose [16] to show that if ν is the uniform distribution on the hypercube and $\ln n \ll nr^2 \ll (\ln n)^d$ then there are $(a_n)_n$ and $(b_n)_n$ such that $(\Delta(G_n) - a_n)/b_n$ tends to a Gumbel distribution, and if $nr^d = \Theta(\ln n)$ then $\Delta(G_n)$ is not finitely concentrated, but the distribution does not look like any of the standard probability distributions.

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A Proof of Lemma 6.4 of Böröczky

Lemma 6.4 above is due to K. Böröczky Jr. and with his kind permission we give a proof here, as it is not readily available from other sources.

Proof of Lemma 6.4: Let $I := \cap_{a \in A}(a + C)$. Then I is compact and convex. We may suppose wlog that $\mathrm{vol}(I) > 0$ (as otherwise there is nothing to prove). Let us remark that

$$I + \mathrm{cl}(-A) \subseteq C.$$

This is because for any $x \in I$ and $a \in A$ there exists a $c \in C$ such that $x = c + a$, by definition of I. Thus, for any $x \in I, a \in A$ we have $x - a \in C$: in other words $I + (-A) \subseteq C$. This also gives that $I + \mathrm{cl}(-A) = \mathrm{cl}(I + (-A)) \subseteq C$ as C is closed.

We now use the Brunn-Minkowski inequality (see chapter 12 of Matousek [11] for a very readable proof). This states that, if $A, B \subseteq \mathbb{R}^d$ are nonempty and compact, then $\mathrm{vol}(A + B) \geq \left(\mathrm{vol}(A)^{\frac{1}{d}} + \mathrm{vol}(B)^{\frac{1}{d}}\right)^d$. By this inequality

$$\mathrm{vol}(I)^{\frac{1}{d}} + \mathrm{vol}(\mathrm{cl}(-A))^{\frac{1}{d}} \leq \mathrm{vol}(I + \mathrm{cl}(-A))^{\frac{1}{d}} \leq \mathrm{vol}(C)^{\frac{1}{d}}.$$

Thus

$$\mathrm{vol}(I) \leq \left(\mathrm{vol}(C)^{\frac{1}{d}} - \mathrm{vol}(A)^{\frac{1}{d}}\right)^d. \tag{82}$$

The lemma will now follow if we show that equality holds in (82) when A is of the form $A = \lambda(-C)$ for some $\lambda > 0$. Let us then suppose that $A = \lambda(-C)$ for some $0 \leq \lambda < 1$ (note that $\lambda \geq 1$ would contradict $\mathrm{vol}(I) > 0$).

We claim that $(1 - \lambda)C \subseteq I$. Let $x \in (1 - \lambda)C$, and let $a \in A = \lambda(-C)$ be arbitrary. We can write $x = (1 - \lambda)c_1$ and $a = -\lambda c_2$ for some $c_1, c_2 \in C$. Because C is convex, $c_3 := (1 - \lambda)c_1 + \lambda c_2 \in C$ and thus $x = a + c_3 \in (a + C)$. As $a \in A$ was arbitrary this gives $x \in I$. Thus indeed $(1 - \lambda)C \subseteq I$, as claimed. But now $\mathrm{vol}(I) \geq (1 - \lambda)^d \,\mathrm{vol}(C)$ and $\mathrm{vol}(A) = \lambda^d \,\mathrm{vol}(C)$); and so equality holds in (82), and we are done. ■